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An efficient index method for the optimal path query over multi-cost networks

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Abstract
In the past couple of decades, graphs have been widely used to model complex relationships among various entities in real applications. Shortest path query is a fundamental problem in graphs and has been well-studied. Most existing approaches for the shortest path problem consider that there is only one kind of cost in networks. However, there always are several kinds of cost in networks and users prefer to select an optimal path under the global consideration of these kinds of cost. In this paper, we study the problem of finding the optimal path in the multi-cost networks. We prove this problem is NP-hard and the existing index techniques cannot be used to this problem. We propose a novel partition-based index with contour skyline techniques to find the optimal path. We propose a vertex-filtering algorithm to facilitate the query processing. We conduct extensive experiments on six real-life networks and the experimental results show that our method has an improvement in efficiency by an order of magnitude compared to the previous heuristic algorithms.

Keywords Index · Optimal path · Multi-cost network

1 Introduction

In the past couple of decades, graphs have been widely used to model complex relationships among various entities in real applications, such as transportation networks, bioinformatics,

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social networks and so on. The shortest path query is a fundamental problem in graphs and has been well studied. For example, in traffic networks, the shortest path query is to find a shortest path between two locations. In social networks, the shortest path query is to find the closest relationships such as friendship between two individuals.

Most existing work about the shortest path problem assume that there is only one kind of cost in the networks. However, the relationships among various entities are always investigated from several distinct aspects. For example, in traffic networks, the paths between two cities are taken into account with several kinds of cost such as road length, toll fee, traffic congestion and so on. It is inadvisable to choose a shortest path only by one kind of cost because the total toll fee of a path with the minimum length may be too expensive to accept for some users. It is important to find an optimal path under global consideration with people's preferences.

A network is called *multi-cost network* if every edge in it has several kinds of cost. Obviously, the shortest path under one kind of cost may not be the optimal path for some users in multi-cost networks. Score function is proposed by user and it can calculate an overall score based on all kinds of cost to measure the optimality for a path. Note that the score functions given by distinct users may be different. Given a score function $f(\cdot)$, a starting vertex v_s and an ending vertex v_e, this paper is to find a path from v_s to v_e with the minimum score and such path is also called an *optimal path* from v_s to v_e under the score function $f(\cdot)$ in the following.

The traditional shortest path problem can be solved by polynomial algorithms e.g., Dijkstra algorithm, and various index techniques are proposed to improve the efficiency. However, these index techniques cannot be used for the optimal path in the multi-cost networks because the score functions given by distinct users may be different. An index built for a score function $f(\cdot)$ cannot cope with the case of another score function $g(\cdot)$. In addition, we prove the optimal path problem is NP-hard in this paper if the score function is non-linear, e.g., $f(x, y) = x^2 + y^2$, and then existing algorithms cannot work under such functions. As discussed in previous studies about transportation networks [3, 7, 13, 14, 26, 29], the non-linear score functions are existent widely and reasonable in real applications. For example, in special conditions such as traffic jam occurring, the traveling time and fuel consumption are nonlinear function, e.g., quadratic function, convex function and so on, with the distance from source to destination [3, 7, 13, 14, 26, 29].

In this paper, we develop a novel partition-based index to find the optimal path in multi-cost networks under various linear or non-linear score functions. The main contributions are summarized below. First, we study the problem of the optimal path recommendation in multi-cost networks and prove it is NP-hard. Second, we propose a partition-based index and contour skyline in the index. We prove the problem of computing contour skyline is NP-hard and give a 2-approximate algorithm. Third, we propose a vertex-filtering algorithm which can filter a large proportion of vertices that cannot be passed through by the optimal path. Finally, we confirm the effectiveness and efficiency of our algorithms using real-life datasets.

The rest of this paper is organized as follows. Section 2 gives the problem statement. Section 3 introduces the partition-based index and how to construct it. Section 4 proposes a vertex-filtering algorithm and discusses how to find the optimal path by partition-based index. We conduct experiments using six real-life datasets in Section 5. The experimental results confirm the effectiveness and efficiency of our approach. Section 6 discusses the related works. We conclude this paper in Section 7.

2 Problem statement

Definition 1 (multi-cost network) A multi-cost network is a simple directed graph, denoted as $G = (V, E, W)$, where V and E are the sets of vertices and edges respectively. W is a set of vectors. Every edge $e \in E$ is represented by $e = (v_i, v_j)$, $v_i, v_j \in V$, and $w(v_i, v_j) \in W$ is the cost vector of (v_i, v_j), $w(v_i, v_j) = (w_1, w_2, \cdots, w_d)$, where w_i is the i-th kind of cost value of edge (v_i, v_j).

In this paper, we assume $w_i \geq 0$. This assumption is reasonable, because the cost cannot be less than zero in real applications. Our work can be easily extended to handle undirected graphs, an undirected edge is equivalent to two directed edges. For simplicity, we only discuss the directed graphs in the following. A path p is a sequence of vertices $(v_0, v_1, \cdots, v_l)$, where $v_i \in V$ and $(v_{i-1}, v_i) \in E$. p is a simple path if and only if there is no repeated vertex in p, i.e., $v_i \neq v_j$, for any $0 \leq i \neq j \leq l$. We use $w(p)$ to denote cost vector of path p, i.e., $w(p) = (w_1(p), w_2(p), \cdots, w_d(p))$, where $w_x(p) = \sum_{i=1}^{l} w_x(v_{i-1}, v_i)$ for $0 \leq x \leq d$.

For a path p in G, a score function is used to calculate an overall score $f(p)$ base on $w(p)$. The score function $f(\cdot)$ is always monotone increasing, i.e., for two different paths p and p', if $(\forall i, w_i(p) \leq w_i(p')) \wedge (\exists i, w_i(p) < w_i(p'))$, then $f(p) < f(p')$. It is a common property and its intuitive meaning is that if all costs of a path p are less those that of p', then the overall score of p must be less than p'. The definition of the optimal path over the multi-cost networks is given below:

Definition 2 (optimal path) Given a multi-cost network G, a score function $f(\cdot)$, a starting vertex v_s and an ending vertex v_e, the optimal path from v_s to v_e, denoted as $p^*_{s,e}$, is a path in G that has the minimum score among all paths from v_s to v_e, i.e., $f(p^*_{s,e}) \leq f(p)$ for any $p \in P_{s,e}$, where $P_{s,e}$ is the set of all simple paths from v_s to v_e.

The following theorem analyzes the complexity of this problem.

Theorem 1 *The problem of finding the optimal path under a non-linear function in the multi-cost networks is NP-hard.*

Proof We reduce the problem of the minimum sum of squares, which is NP-complete [9], to this problem. The minimum sum of squares problem is as follows. Given a number set $A = \{a_1, a_2, \cdots, a_n\}$ of size n and an integer $k \leq |A|$, find a partition $\mathcal{A}^* = \{A_1, A_2, \cdots, A_k\}$ of A such that $\sum_{j=1}^{k}(\sum_{a_i \in A_j} a_i)^2$ is minimum. Note that A_j $(1 \leq j \leq k)$ cannot be an empty set for an optimal partition $\mathcal{A}^*$. Given an instance of the minimum sum of squares problem, it can be converted to an instance of the optimal path problem shown in Figure 1. We create a graph G with $n + 1 + kn$ vertices, $\{v_1, v_2, \cdots, v_{n+1}\} \cup \{v_{i,j} | 1 \leq i \leq n, 1 \leq j \leq k\}$. Here, $v_{i,j} (1 \leq j \leq k)$ is placed between v_i and v_{i+1}. We create the edges in G as follows. For $\forall 1 \leq i \leq n$ and $\forall 1 \leq j \leq k$, we create an edge $e_{i,(i,j)}$ from v_i to $v_{i,j}$. The cost of edge $e_{i,(i,j)}$ is assigned as $w(e_{i,(i,j)}) = (0, \cdots, 0, \frac{a_i}{2}, 0, \cdots, 0)$, i.e., the j-th cost value of $w(e_{i,(i,j)})$ is $\frac{a_i}{2}$ and the others are zero. Similarly, we create an edge $e_{(i,j),i+1}$ from $v_{i,j}$ to v_{i+1}. The cost of edge $e_{(i,j),i+1}$ is also $w(e_{(i,j),i+1}) = (0, \cdots, 0, \frac{a_i}{2}, 0, \cdots, 0)$, i.e., the j-th cost value of $w(e_{(i,j),i+1})$ is $\frac{a_i}{2}$ and the others are zero. Let $v_1 = v_s$ and $v_{n+1} = v_e$. Score function is $f(w_1, \cdots, w_k) = \sum_{i=1}^{k}(w_i)^2$. Here, $(w_1, \cdots, w_k)$ is the cost vector $w(p)$ of a path p. Obviously, if a path p travels through an edge $e_{i,(i,j)}$, it must travel through $e_{(i,j),i+1}$. We can concatenate $e_{i,(i,j)}$ and $e_{(i,j),i+1}$ as a new edge $e^j_{i,i+1}$

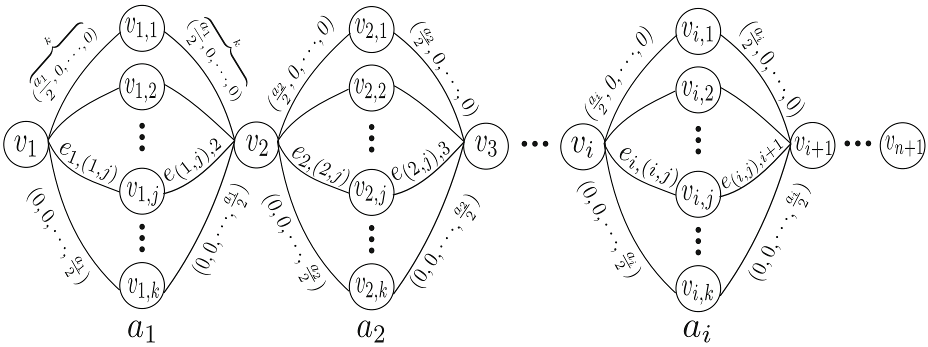


Figure 1 A diagram of Theorem 1

from v_i to v_{i+1}. $e^j_{i,i+1}$ is called the j-th edge from v_i to v_{i+1} in G. The cost of $e^j_{i,i+1}$ is $(0, \cdots, 0, a_i, 0, \cdots, 0)$, i.e., the j-th cost value of $w(e^j_{i,i+1})$ is a_i and the others are zero. For any path p from v_s to v_e in graph G, the j-th cost value $w_j(p)$ of $w(p)$ is equal to the sum of the j-th cost values of all the edges in p. Let E^j_p be the set of all the j-th edges in G that p travels through, i.e., $E^j_p = \{e^j_{i,i+1} | e^j_{i,i+1} \in p, 1 \leq i \leq n\}$. Then $\{E^j_p | 1 \leq j \leq k\}$ corresponds to a partition $\mathcal{A} = \{A_j | 1 \leq j \leq k\}$ of A, where A is the number set $\{a_1, a_2, \cdots, a_n\}$ and A_j $(1 \leq j \leq k)$ is the number set of the j-th cost value of all the edges in E^j_p, i.e., $A_j = \{w_j(e) | e \in E^j_p\}$. Consequently, an optimal path p^* with the minimum score corresponds to an optimal partition $\mathcal{A}^*$ for A such that $\sum_{j=1}^{k}(\sum_{a_i \in A_j} a_i)^2$ is the minimum. Note that this reduction is in polynomial time. If we find an optimal path from v_s to v_e in G in polynomial time, then we also can find an optimal partition $\mathcal{A}^*$ for number set A. Therefore, the problem of finding the optimal path over the multi-cost graphs is NP-hard. □

We use the following example to illustrate how to convert a problem of the minimum sum of squares to the optimal path problem.

Given a integer set $A = \{1, 2, 3\}$ and $k = 2$, the minimum sum of squares problem is to find a partition $A^* = \{A_1, A_2\}$ of A to minimize $\sum_{j=1}^{k} \left(\sum_{a_i \in A_j} a_i\right)^2$. By constructing the graph G shown in Figure 2, this problem can be converted to an optimal path problem

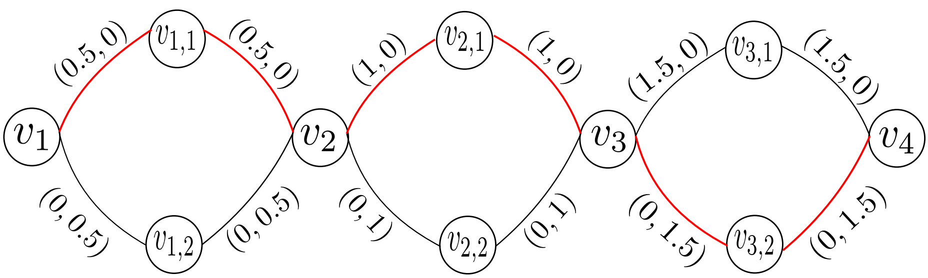


Figure 2 An example of Theorem 1

under the function $f(w_1, \cdots, w_k) = \sum_{i=1}^{k} (w_i)^2$. Note that the optimal path p^* is $v_1 \to v_{1,1} \to v_2 \to v_{2,1} \to v_3 \to v_{3,2} \to v_3$, which is shown by red line in Figure 2. The cost of the optimal path is $(1 + 2, 3)$ and then 1 and 2 are taken into a set and 3 is taken into another set. Therefore, it corresponds to an optimal partition $A^* = \{\{1, 2\}, \{3\}\}$, which is exactly the soluion for the minimum sum of squares problem on integer set $A = \{1, 2, 3\}$.

If the score function is linear, e.g., $f(x, y) = x + y$, this problem can be solved in polynomial time by Dijkstra-based algorithms. However, the existing index techniques for traditional shortest path problem cannot be used for this problem even though the score function is linear. It is because the score functions given by distinct users may be different and then an index built for a score function $f(\cdot)$ cannot cope with the case of another score function $g(\cdot)$. There are some works [8] about the optimal path problem under linear score funtions and propose some query optimization techniques, e.g., as Contraction Hierarchies. However, such techniques cannot be used for non-linear functions because it is extremely complicated to calculate hyperplane for arbitrary non-linear functions. On the other hand, if the score function is non-linear, e.g., $f(x, y) = x^2 + y^2$, this problem is NP-hard and only a small number of heuristic algorithms are proposed to solve it [31]. In this paper, we develop a novel partition-based index to find the optimal path in multi-cost networks and it can improve the querying efficiency significantly for the heuristic algorithms under the non-linear score functions. Note that our partition-based index also can handle the case of the linear score functions, but it may not be better than the index that elaborately developed for the linear functions. Therefore, our partition-based index is more appropriate for the optimal path query under non-linear functions.

3 Partition-based index

3.1 What is the partition-based index?

Given a graph $G(V, E)$, a k-partition of G is a collection $\{V_1, \cdots, V_k\}$ satisfying the following conditions: (1) every V_p is a subset of V; (2) for $\forall V_p, V_q$ $(p \neq q)$, $V_p \cap V_q = \emptyset$; (3)$V = \bigcup_{1 \le p \le k} V_p$. A vertex v_i is called an ***entry (or exit)*** of V_p, if (1) $v_i \in V_p$; and (2) $\exists v_j$, $v_j \notin V_p \wedge v_j \in N^-(v_i)$ (or $v_j \in N^+(v_i)$), where $N^-(v_i)$ and $N^+(v_i)$ are v_i's incoming and outgoing neighbor set respectively. Entries and exits are also called the ***border vertices***. We use $V_p.entry$ and $V_p.exit$ to denote the entry set and exit set of V_p, and use $V.entry$ and $V.exit$ to denote the sets of all entries and exits in G, respectively. Obviously, $V.entry = \bigcup_{1 \le p \le k} V_p.entry$ and $V.exit = \bigcup_{1 \le p \le k} V_p.exit$.

A partition-based index includes two parts: ***inter-index*** and ***inner-index***. We first introduce the *lower bound of optimal path* (LBOP) and *skyline path*.

For a multi-cost network G with d kinds of cost, $\mathcal{G}_x$ $(1 \le x \le d)$ is a weighted graph with the same structure as G, and the weight of every edge (v_i, v_j) in $\mathcal{G}_x$ is the x-th cost $w_x(v_i, v_j)$ of $w(v_i, v_j)$. For any two vertices $v_i, v_j \in G$, $\mathcal{P}_{i,j} = \{p^1_{i,j}, \cdots, p^d_{i,j}\}$ is *the set of single-one cost shortest paths* from v_i to v_j, where $p^x_{i,j}$ is the shortest path from v_i to v_j in $\mathcal{G}_x$. We use $\phi^x_{i,j}$ to denote the weight of $p^x_{i,j}$. The cost vector $\Phi_{i,j} = (\phi^1_{i,j}, \cdots, \phi^d_{i,j})$ is called the ***lower bound of the optimal path*** (LBOP) from v_i to v_j in G.

Let p and p' be two different paths in a multi-cost graph G. We say p dominate p', denoted as $p \prec p'$, iff for $\forall i$ $(1 \le i \le d)$, $w_i(p) \le w_i(p')$, and $\exists i$ $(1 \le i \le d)$, $w_i(p) < w_i(p')$. Here, $w_i(p)$ and $w_i(p')$ are the i-th cost value of $w(p)$ and $w(p')$, respectively. For two vertices $v_i, v_j \in G$, a path p is a ***skyline path*** from v_i to v_j iff p cannot be dominated by any other path p' from v_i to v_j.

For any path $p_{i,j}$ from v_i to v_j, the cost vector of $p_{i,j}$ is $w(p_{i,j}) = (w_1(p_{i,j}), \cdots, w_d(p_{i,j}))$, then we have $\Phi_{i,j} \preccurlyeq p_{i,j}$, i.e., for $\forall x$ $(1 \leq x \leq d)$, $\phi^x_{i,j} \leq w_x(p_{i,j})$.

Lemma 1 guarantees that $\Phi_{i,j}$ is the strict lower bound of the optimal path from v_i to v_j in multi-cost network G.

Lemma 1 *$\Phi_{i,j}$ is the strict lower bound for the optimal path from v_i to v_j in G, that is, there does not exist another lower bound $\Phi'_{i,j}$ such that $\Phi_{i,j} \prec \Phi'_{i,j}$ and $\Phi'_{i,j} \preccurlyeq p_{i,j}$ for any path $p_{i,j}$ from v_i to v_j.*

Proof We prove it by contradiction. Assume that there is $\Phi'_{i,j}$ satisfying $\Phi_{i,j} \prec \Phi'_{i,j}$, then $\exists x$ $(1 \leq x \leq d)$, such that $\phi'^x_{i,j} > \phi^x_{i,j}$. On the other hand, because $p^x_{i,j}$ is a path from v_i to v_j and then $\Phi'_{i,j} \preccurlyeq p^x_{i,j}$. It means $\phi'^x_{i,j} \leq \phi^x_{i,j}$, which is a contradiction. □

It is obvious that every individual x-th weight $\phi^x_{i,j}$ of $\Phi_{i,j}$ is exactly the shortest distance from v_i to v_j under considering the x-th cost as a single cost in graph G. If $\Phi_{i,j}$ is not the strict lower bound of the optimal path from v_i to v_j, there must exist a $\phi^x_{i,j}$ which is larget than the shortest distance from v_i to v_j under the x-th cost, which is contradictory to the definition of $\Phi_{i,j}$.

Inter-index Inter-index is essentially a matrix A to maintain the LBOP for every pair of border vertex and entry in G. Each row represents a border vertex (entry or exit) v_i and each column represents an entry v_j in G. The size of A is $(|V.exit| + |V.entry|) \times |V.entry|$. Each cell $A_{i,j}$ includes two elements: $\Phi_{i,j}$ and $\mathcal{P}_{i,j}$.

Inner-index Inner-index consists of k sub-indexes and every sub-index I_p is associated with a vertex subset V_p. I_p includes two parts: (i) *Skyline-Path-Inner-Index* I^S_p; and (ii) LBOP-*Inner-Index* I^L_p.

Skyline-Path-Inner-Index I^S_p of V_p is a collection of skyline path sets for all pairs of entry and exit in V_p, i.e., $I^S_p = \{SP_{(i,j);p} | v_i \in V_p.entry, v_j \in V_p.exit\}$. $SP_{(i,j);p}$ is the set of all skyline paths from v_i to v_j in G_p, where G_p is the induced subgraph of V_p on G. Note that the paths in $SP_{(i,j);p}$ only pass through the vertices in V_p.

LBOP-Inner-Index I^L_p of V_p is essentially a matrix M_p of size $|V_p| \times |V_p|$ to maintain LBOPs for all pairs of vertices v_i and $v_j \in V_p$. Actually, we only need to maintain a smaller matrix M'_p as I^L_p in memory. M'_p is a sub-matrix of M_p. It maintains all the LBOPs from an entry to a vertex in V_p and all the LBOPs from a vertex to an exit in V_p. The remaining sub-matrix $M^-_p = M_p \setminus M'_p$ $(1 \leq p \leq k)$ is maintained in the disk. M^-_s and M^-_e are taken into the memory when the starting vertex v_s and the ending vertex v_e are given.

By inter-index and LBOP-inner-index, $\Phi_{i,j}$ can be calculated easily for any pair of vertices v_i and v_j in G. Given a starting vertex v_s and an ending vertex v_e, we use V_s and V_e to denote the vertex subsets including v_s and v_e respectively. If $V_s = V_e$, we can obtain $\Phi_{s,e}$ from LBOP-inner-index I^L_p directly. If $V_s \neq V_e$, we calculate $\Phi_{s,e}$ by Lemma 2.

Lemma 2 *Given two vertices v_s and v_e in a multi-cost network G, V_s and V_e are two distinct vertex subsets including v_s and v_e respectively. Let v_i be an entry of V_e. Thus for $\forall x$ $(1 \leq x \leq d)$, we have $\phi^x_{s,e} = \min\{\phi^x_{s,i} + \phi^x_{i,e} | v_i \in V_e.entry\}$, where $\phi^x_{s,e}$, $\phi^x_{s,i}$ and $\phi^x_{i,e}$ are the x-th cost of* LBOP *$\Phi_{s,e}$, $\Phi_{s,i}$ and $\Phi_{i,e}$ respectively.*

Proof We know $\phi^x_{s,e}$ $(1 \le x \le d)$ is the weight of the shortest path $p^x_{s,e}$ in network $\mathcal{G}_x$, which must pass through an entry v_i in $V_e.entry$. Therefore, $p^x_{s,e}$ can be considered as two parts: (1) sub-path from v_s to v_i; and (2) sub-path from v_i to v_e. Because $\phi^x_{s,i}$ and $\phi^x_{i,e}$ are the shortest distance from v_s to v_i and from v_i to v_e in $\mathcal{G}_x$ respectively, then $\phi^x_{s,i} + \phi^x_{i,e} \le \phi^x_{s,e}$. On the other hand, $\phi^x_{s,e}$ is the minimum among all the paths from v_s to v_e, thus $\phi_{(s,e)^x} \le \phi^x_{s,i} + \phi^x_{i,e}$, then we have $\phi^x_{s,e} = \phi^x_{s,i} + \phi^x_{i,e}$. Next, we prove that v_i is exactly the entry minimizing $\phi^x_{s,i} + \phi^x_{i,e}$. It is obvious otherwise $p^x_{s,e}$ is not the shortest path in $\mathcal{G}_x$. It means $\phi^x_{s,e} = \min\{\phi^x_{s,i} + \phi^x_{i,e} | v_i \in V_e.entry\}$. □

Lemma 2 shows how to calculate LBOP for two vertices v_s and v_e in distinct vertex subsets V_s and V_e respectively.

$\Phi_{s,e}$ can be computed by $\Phi_{s,i}$ and $\Phi_{i,e}$ for all entries v_i in V_e. The i-th value $\phi^x_{s,e}$ of $\Phi_{s,e}$ equals to the shortest distance from v_s to v_e via v_i according to the i-th cost value on every edge, where v_i is an entry minimize such shortest distance.

$\Phi_{s,e}$ can be calculated in two cases: (1) $v_s \in V_s.entry \cup V_s.exit$; and (2) $v_s \notin V_s.entry \cup V_s.exit$. For case (1), $\phi^x_{s,i}$ and $\phi^x_{s,i}$ can be directly retrieved from inter-index and LBOP-inner-index I^L_e respectively. Therefore, the minimum value of $\phi_{(s,i)^x} + \phi_{(i,e)^x}$ can be easily calculated as $\phi^x_{s,e}$ by Lemma 2. For case (2), because $\phi^x_{s,i}$ is not maintained in inter-index, it is necessary to calculate the minimum value of $\phi^x_{s,j} + \phi^x_{j,i} | v_j \in V_s.exit\}$ as $\phi^x_{s,i}$ and then calculate $\phi^x_{s,e}$ in a similar way as the case (1). The algorithm to compute $\Phi_{s,e}$ for any two vertices v_s and v_e in G is shown in Algorithm 1. The set $\mathcal{P}_{s,e}$ of the single-one cost shortest paths can be calculated in the similar way as calculating $\Phi_{s,e}$.

Algorithm 1 COMPUTE-LBOP(I, s, t).

Input: index I, starting vertex v_s and ending vertex v_e
Output: LBOP$\Phi_{s,e}$ from v_s to v_e.

```
if V_s = V_e then
    return Φ_{s,e} from I_s^L (or (I_e^L));
else
    if v_s ∈ V_s.entry ∪ V_s.exit then
        PROCEDURE(v_s, v_e, V_e.entry);
    else
        for v_i ∈ V_e.entry do
            PROCEDURE(v_s, v_i, V_s.exit);
        PROCEDURE(v_s, v_e, V_e.entry);
    return Φ_{s,e};
```

Algorithm 2 PROCEDURE(v_i, v_j, V).

```
for x = 1 to d do
    for each v_r ∈ V do
        φ* ← φ_{(i,r);x} + φ_{(r,j);x};
        if φ_{(i,j);x} > φ* then
            φ_{(i,j);x} ← φ*;
```

3.2 How to construct partition-based index?

3.2.1 Inter-index and LBOP-inner-index

For LBOP-inner-index I_p^L of vertex subset V_p, the shortest path algorithms can be used to calculate $\Phi_{i,j}$ for every pair of vertex v_i and v_j in V_p. For inter-index, $\Phi_{i,j}$ for every pair of border vertex $v_i \in V.entry \cup V.exit$ and entry $v_j \in V.entry$ also can be calculated by the shortest path algorithms. It worth noting that it is not necessary to maintain $\Phi_{i,j}$ in inter-index if v_i and v_j are in the same vertex subset V_p because it has been maintained in the LBOP-inner-index.

3.2.2 Skyline-path-inner-index

For every I_p^S in Skyline-path-inner-index, $I_p^S = \{SP_{(i,j);p} | v_i \in V_p.entry, v_j \in V_p.exit\}$, it is necessary to calculate $SP_{(i,j);p}$ for every pair of entry v_i and exit v_j in V_p. Note that $SP_{(i,j);p}$ is the set of all the skyline paths from v_i to v_j in V_p. A skyline path from v_i to v_j is also called a pareto path. In the past decades, several works [5, 6, 15–17, 19, 31] have been studied the problem of computing the skyline paths or pareto paths and these methods can be used to compute skyline paths or pareto paths for building the Skyline-path-inner-index. In this paper, we use the heuristic algorithm proposed in [31] to compute skyline paths for the Skyline-path-inner-index construction.

3.3 Contour skyline set

Given a skyline-path-inner-index I_p^S, each skyline path $p \in SP_{(i,j);p}$ can be regarded as a skyline point p in the d-dimensional space according to $w(p)$. Note that there may be some skyline points are close to each other and this property is helpful for improving the efficiency of the optimal path query. In this section, we propose the definition of the contour skyline set. All skyline points in $SP_{(i,j);p}$ can be partitioned into several groups and each group corresponds to a contour skyline point. We compute a ***contour skyline point*** for every group and the set of the contour skyline points is called the ***contour skyline set*** of $SP_{(i,j);p}$.c In query processing, if the contour skyline point can be pruned, then all the skyline paths in this group can be filtered, which makes query processing more efficient.

Figure 3 is an example of the contour skyline set in the cluster V_p. $p_1, \cdots, p_9$ are the skyline points in a 2-dimensional space and each p_i is a skyline path p_i. We observe that $R_1 = \{p_1, p_2, p_3\}$, $R_2 = \{p_4, p_5, p_6, p_7\}$ and $R_3 = \{p_8, p_9\}$ are three groups such that the skyline points in the same group are space proximity. Then cp_1, cp_2 and cp_3 are the contour skyline points corresponding to R_1, R_2 and R_3 respectively. Let $w(cp_i) = (w_1(cp_i), w_2(cp_i))$ be the cost vector of cp_i. It is obvious that cp_i is the LBOP of the skyline paths in R_i, i.e., $w_x(cp_i) = \min\{w_x(p) | p \in R_i\}$, where $w_x(cp_i)$ and $w_x(p)$ are the x-th cost value of $w(cp_i)$ and $w(p)$ respectively. Therefore, the problem to compute the contour skyline points is equivalent to partition the skyline points into several different groups such that the points in each group are more space proximity. Given a specified r, our goal is to partition the skyline points into r groups. To do that, we introduce the concept of the diameter for such a group. For a group R_i, the diameter of R_i, denoted as $\mathcal{D}(R_i)$, is defined as the maximum Euclidean distance among all the pairs of the points in S. Formally,

$$\mathcal{D}(R_i) = max\{\mathsf{dist}(p, p') | p, p' \in R_i\} \quad (1)$$

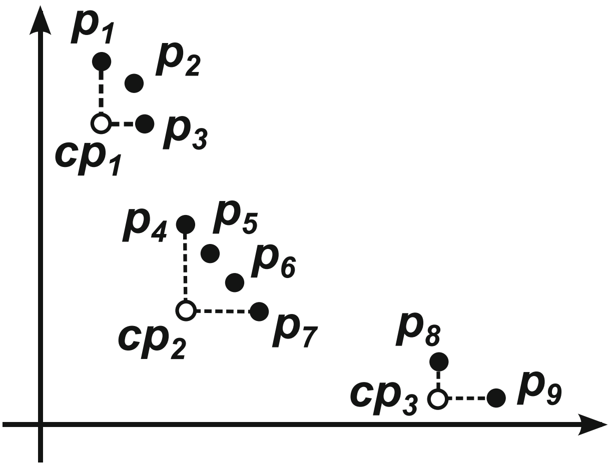


Figure 3 An example of contour skyline set

where, $\mathsf{dist}(p, p')$ is the Euclidean distance between p and p' in the multi-dimensional space. Given a r-partition $\mathcal{R} = \{R_1, \cdots, R_r\}$, we define the diameter $\mathcal{D}(\mathcal{R})$ of $\mathcal{R}$ below:

$$\mathcal{D}(\mathcal{R}) = max\{\mathcal{D}(R_i)|R_i \in \mathcal{R}\} \tag{2}$$

Intuitively, $\mathcal{D}(\mathcal{R})$ quantifies the partition quality as the maximum distance between any two points in the same group. A partition $\mathcal{R}$ is good if, for every two points in the same group, they are close to each other.

Definition 3 (Contour skyline) Given two vertices v_x and v_y in vertex subset V_p, $SP_{(x,y);p}$ is the skyline path set from v_x to v_y in the induced subgraph G_p, every path in $SP_{(x,y);p}$ is a skyline point in d-dimensional space. Given an integer r, an optimal r-partition $\mathcal{R}_{opt}$ is a partition to minimize $\mathcal{D}(\mathcal{R})$. For every group R_i in $\mathcal{R}_{opt}$, the *contour skyline point* cp_i is the LBOP of the skyline paths in R_i, the set of all cp_i is called the *contour skyline set* of $SP_{(x,y);p}$, denoted as $CS_{(x,y);p}$.

The efficiency of the optimal path query can be improved by $CS_{(x,y);p}$. We introduce it in Section 4.2. Next, we discuss how to compute the contour skyline points. This problem is to find the optimal partition $\mathcal{R}_{opt}$ for all the skyline points in $SP_{(x,y);p}$. In the case of 2D space, we propose a dynamic programming method to compute the optimal partition $SP_{(x,y);p}$. We prove this problem is NP-hard in 3D or higher dimensional space. We give a 2-approximate algorithm and show there is no $(2-\epsilon)$-approximate solution in the polynomial time.

Case 1 (2D space) Assume that $SP_{(x,y);p}$ has been already computed and let m be the size of $SP_{(x,y);p}$. We use $S = \{p_1, \cdots, p_m\}$ to denote the set of all skyline points in $SP_{(x,y);p}$, where all p_i in S are sorted in ascending order of their x-coordinates. We use S_i to denote $\{p_1, p_2, \cdots, p_i\}$. Specially, $S_0 = \emptyset$. We also use a notation $opt(i, t)$ to denote the optimal t-partition for S_i. Obviously, the optimal r-partition $\mathcal{R}_{opt}$ for S is essentially $opt(m, r)$. Let

$S_{j,i}$ be the point set $\{p_j, \cdots, p_i\}$, where $0 \leq j \leq i \leq m$. Then we have the following recursive equation:

$$\mathcal{D}(opt(i,t)) = \min_{j=t-1}^{i} \{\max\{\mathcal{D}(opt(j-1,t-1)), \mathcal{D}(S_{j,i})\}\} \tag{3}$$

The meaning of (3) is that: without loss generality, assume that the optimal t-partition of S_i is $\{R_1, \cdots, R_t\}$, where R_t is the last group which consists of $\{p_j, \cdots, p_i\}$. Then, $\{R_1, \cdots, R_{t-1}\}$ must be the optimal $(t-1)$-partition for S_{j-1}. Let $j_{\min}$ be the value of j minimizing (3), then we have

$$\begin{aligned} opt(i,t) &= opt(j_{\min}-1, t-1) \cup S_{j_{\min},i} \\ opt(i,1) &= S_i \end{aligned} \tag{4}$$

By (3) and (4), a dynamic programming method can be utilized to compute the optimal r-partition for $SP_{(x,y);p}$ in 2D space.

Case 2 (3D and the higher dimensional space) In 3D and the higher dimensional space , we prove the optimal r-partition problem is NP-hard by reducing the r-split problem in 2D space, which is NP-hard, to this problem. Given a set of points $\{p_1, \cdots, p_n\}$ in 2D space, the r-split problem is to find a set of r groups $\{B_1, \cdots, B_r\}$ that minimizes

$$\max_{1 \leq x \leq r} \{\max\{dist(p_i, p_j) | p_i, p_j \in B_x\}\} \tag{5}$$

This problem is similar to the r-partition problem for the skyline points, but when the points in space are the skyline points, the complexity of the r-split problem is unknown. We give Lemma 3 as follows:

Lemma 3 *For dimensionality $d \geq 3$, the r-partition problem is NP-hard.*

Proof Given a set of points $\{p_1, \cdots, p_n\}$ in 2D space, we map each of them to a skyline point in 3D space. For a point p_i with x-coordinate $p_i(x)$ and y-coordinate $p_i(y)$, it is mapped to a point p'_i in 3D space with x, y and z-coordinates: $p'_i(x) = -\frac{1}{\sqrt{2}} p_i(x) + \frac{1}{2} p_i(y)$, $p'_i(y) = \frac{1}{\sqrt{2}} p_i(x) + \frac{1}{2} p_i(y)$, and $p'_i(z) = -\frac{1}{\sqrt{2}} p_i(y)$. For any two points in 3D space p'_1 and p'_2, if $p'_1(x) > p'_2(x)$ and $p'_1(y) > p'_2(y)$, then $p'_1(z) < p'_2(z)$. It means each point in 3D space is a skyline point. On the other hand, we also find $dist(p'_1, p'_2) = dist(p_1, p_2)$, where $dist(p_i, p_j)$ is the Euclidean distance between p_i and p_j. This reduction is in the polynomial time. If we can find the optimal r-partition in the polynomial time, then we can solve r-split problem in the polynomial time.

Given a set S of points in 3D space, we can convert it to a d-dimensional point set S' for any $d \geq 3$ easily. We assign $(d-3)$ zeros to all the other coordinates for any point in S. The optimal r-partition for S' is obviously the optimal r-partition for S in 3D space. It is in the polynomial time for the reduction from 3D space to the d-dimensional space. $\square$

We give a greedy algorithm for r-partition on a given $SP_{(x,y);p}$ in a vertex subset V_p. The main idea is as follows: In the initialization phase, all the points are assigned to a group R_1. One of these points, denoted as bp_1, is selected as the "base point" of R_1. The selection of bp_1 is arbitrary. During each iteration, some points in $R_1, \cdots, R_j$ are moved into a new group R_{j+1}. Also, one of these points will be selected as the "base point" of the new group, i.e., bp_{j+1}. The construction of the new group is accomplished by first finding a point p_i, in one of the previous j groups $\{R_1, \cdots, R_j\}$, whose distance to the base point of group it belongs is maximal. Such a point will be moved into the group R_{j+1} and selected as the "base point" of R_{j+1}. A point in any of the previous groups will be moved into group R_{j+1} if its distance to p_i is not larger than the distance to the base point of group it belongs to. With the r-partition, the $CS_{(x,y);p}$ of $SP_{(x,y);p}$ can be computed easily according to the definition of the contour skyline set.

This algorithm is guaranteed as a 2-approximate solution because there is no $(2-\epsilon)$-approximate solution in the polynomial time if $P \neq NP$, as analysis in [12].

In summary, for each $SP_{(x,y);p}$ in vertex subset V_p, we compute the contour skyline set $CS_{(x,y);p}$. We also maintain every $CS_{(x,y);p}$ in I_p^S.

3.4 How to partition graph to K vertex subsets

For optimal path problem in the multi-cost networks, the less number of edges among different vertex subsets results in the less number of entries and exits in the multi-cost network, and then the size of partition-based index becomes smaller. The objective of the partition is to make the edges dense in the same vertex subset and sparse among different vertex subsets. It is an optimal partition problem and has been well studied in the past couple of decades [1]. In this paper, we use the classic multi-level graph partitioning algorithm, proposed by Metis et al. in [1], to partition the networks in experiments.

3.5 Time and space complexity

Because the shortest path problem and skyline path problem have been studied well in past decades and there are several efficient methods to solve these two problems, we use $\alpha(G)$ and $\beta(G)$ to denote the time complexity of calculating the shortest path and the skyline path set respectively on a graph G. As our discussion in Section 3.2, it needs $O(d\alpha(G)|V.border| \times |V.entry|)$ and $O(d\alpha(G)\sum_{p=1}^{k}|V_p| \times |V_p.border|)$ time to compute inter-index and LBOP-inner-index respectively. Because for every pair of entry and exit in V_p, the time complexity of computing the contour skyline set is not larger than computing skyline paths, then it needs $O(\beta(G_p)\sum_{p=1}^{k}|V_p.entry| \times |V_p.exit|)$ time to compute skyline-inner-index. Therefore, the time complexity for the partition-based index construction is $O(d\alpha(G)(|V.border| \times |V.entry| + \sum_{p=1}^{k}|V_p|$
$\times|V_p.border|) + \beta(G_p)\sum_{p=1}^{k}|V_p.entry| \times |V_p.exit|)$.

Because the LBOPs are maintained in inter-index and LBOP-inner-index as two Matrices with size $|V.border| \times |V.entry|$ and $|V_p| \times |V_p.border|$ respectively, and the skyline paths are maintained in skyline-inner-index for every pair of entry and exit in V_p, then the space complexity of the partition-based index is $O(d(|V.border| \times |V.entry| + \sum_{p=1}^{k}|V_p| \times |V_p.border|) + s\sum_{p=1}^{k}|V_p.entry| \times |V_p.exit|)$, where s is the averge number of the skyline path for a pair of entry and exit.

4 Query processing

Given a multi-cost network $G(V, E, W)$, a starting vertex v_s and an ending vertex v_e, V_s and V_e are the vertex subsets including v_s and v_e respectively. A shrunk graph $\bar{G} = (\bar{V}, \bar{E})$ can be derived from partition-based index. $\bar{V}$ consists of three sets: (1) V_s; (2) V_e, and (3) $\bigcup_{p \neq s,e}(V_p.entry \cup V_p.exit)$. The edges in $\bar{E}$ satisfy three following conditions: (1) $(v_i, v_j) \in \bar{E}$, iff $((v_i, v_j) \in E) \wedge ((v_i, v_j \in V_s) \vee (v_i, v_j \in V_e))$; (2) $(v_i, v_j) \in \bar{E}$, iff $((v_i, v_j) \in E) \wedge ((v_i \in V_p.exit) \wedge (v_j \in V_q.entry))$, where $V_p \neq V_q$; and (3) m edges $\{(v_i, v_j)^1, \cdots, (v_i, v_j)^m\}$ are constructed for any pair of entry v_i and exit v_j in V_p, where $V_p \neq V_s$ and $V_p \neq V_e$. Note that m is the size of $SP_{(i,j);p}$. In case (3), every edge $(v_i, v_j)^\alpha (1 \leq \alpha \leq m)$ from v_i to v_j represents a skyline path in $SP_{(i,j);p}$. The following theorem guarantees the optimal path problem on $G(V, E)$ is equivalent to that on $\bar{G}(\bar{V}, \bar{E})$.

Theorem 2 *Given a multi-cost graph $G(V, E)$, a starting vertex v_s and an ending vertex v_e on G, a shrunk graph $\bar{G}(\bar{V}, \bar{E})$ regarding v_s and v_e can be constructed. Finding the optimal path from v_s to v_e in G is equivalent to finding the optimal path from v_s to v_e in $\bar{G}$.*

Proof First, we prove that an optimal path p from v_s to v_e in G is also an optimal path in $\bar{G}$. p must be a path from v_s to v_e in $\bar{G}$, otherwise some part of p can be dominated by a skyline path in a cluster. A new path can be constructed by using this skyline path instead of this part in p. By the monotonicity of the score function $f(\cdot)$, the score of the new path is less than the score of p, which contradicts with that p is the optimal path in G. Moreover, p must be an optimal path from v_s to v_e in $\bar{G}$, otherwise there must exist another path p' whose score is less than p in $\bar{G}$. Obviously, p' is also a path in G, thus it contradicts with that p is the optimal path in G.

Next, we prove that an optimal path p in $\bar{G}$ is also an optimal path in G. Assume that there exists another path p' whose score is less than p in G, we consider two cases. First, p' is also a path in $\bar{G}$, then p is not the optimal path in $\bar{G}$ because p''s score is less than p's score. Second, p' is not a path in $\bar{G}$, then p' must be dominated by another path p'' in $\bar{G}$ and the score of p'' is less than the score of p in $\bar{G}$. It contradicts with that p is the optimal path in $\bar{G}$. □

Algorithm 3 VERTEX-FILTERING $(\bar{G}(\bar{V}, \bar{E}), v_s, v_e, f(\cdot))$.

Input: $\bar{G}(\bar{V}, \bar{E})$, the score function $f(\cdot)$, the starting vertex v_s and the ending vertex v_e;
Output: the optimal path $p^*_{s,e}$.

1: $\tau \leftarrow \min\{f(p^x_{s,e} | p^x_{s,e} \in \mathcal{P}_{s,e}\}$;
2: **for** each $v_i \in \bar{V}$ **do**
3:     **if** $\tau < f(\Phi_{s,i} + \Phi_{i,e})$ **then**
4:         $\bar{V} \leftarrow \bar{V} - \{v_i\}$;
5: OPTIMAL-PATH $(\bar{G}(\bar{V}), v_s, v_e, f(\cdot))$
6: **return** $p^*_{s,e}, \tau$;

Based on Theorem 2, the optimal path from v_s to v_e on $G(V, E)$ is equivalent to the optimal path on $\bar{G}(\bar{V}, \bar{E})$. The process of finding the optimal path includes two steps: (1) vertex-filtering; and (2) query processing.

4.1 Vertex-filtering

We propose a vertex-filtering algorithm which can effectively filter vertices from $\bar{G}(\bar{V}, \bar{E})$. Given two vertices v_i and v_j in $\bar{G}$, $\Phi_{i,j}$ and $\mathcal{P}_{i,j}$ can be calculated by Algorithm 1. Obviously, $\tau = \min\{f(p^x_{s,e}) | p^x_{s,e} \in \mathcal{P}_{s,e}\}$ is an upper bound of the score of the optimal path from

v_s to v_e. If $\mathcal{P}_{s,e} = \emptyset$, then there does not exist a path from v_s to v_e and algorithm immediately return $p^*_{s,e} = \emptyset$. For any v_i in $\bar{G}$, if $\tau < f(\Phi_{s,i} + \Phi_{i,e})$, then v_i can be removed from $\bar{G}$. In the other words, the optimal path from v_s to v_e cannot pass through v_i. Theorem 3 guarantees the correctness of the vertex filtering.

Theorem 3 *Given a multi-cost graph $G(V, E)$, a score function $f(\cdot)$, a starting vertex v_s and an ending vertex v_e, a shrunk graph $\bar{G}(\bar{V}, \bar{E})$ can be constructed. $\mathcal{P}_{s,e}$ is the set of the single-one cost shortest paths from v_s to v_e, $\mathcal{P}_{s,e} \neq \emptyset$. τ is an upper bound of the optimal path from v_s to v_e, $\tau = \min\{f(p^x_{s,e})|p^x_{s,e} \in \mathcal{P}_{s,e}\}$. For any vertex v_i in $\bar{G}$, if $\tau < f(\Phi_{s,i} + \Phi_{i,e})$, where $\Phi_{s,i}$ and $\Phi_{i,e}$ are the* LBOP *from v_s to v_i and the* LBOP *from v_i to v_e respectively, then the optimal path from v_s to v_e cannot travel through v_i.*

Proof We only need to prove that, for any path p traveling through v_i, there exists a path p' without traveling through v_i, such that $f(p') < f(p)$. Obviously, p consists of two segments: (i) the sub-path $p_{s,i}$ from v_s to v_i; and (ii) the sub-path $p_{i,e}$ from v_i to v_e. By the definition of the LBOP, we have $\Phi_{s,i} \preccurlyeq p_{s,i}$ and $\Phi_{i,e} \preccurlyeq p_{i,e}$. Thus, $\Phi_{s,i} + \Phi_{i,e} \preccurlyeq p$. By the monotonicity of the score function $f(\cdot)$, $f(\Phi_{s,i} + \Phi_{i,e}) \leq f(p)$. Let p' be the path in $\mathcal{P}_{s,e}$ whose score is τ, i.e., $f(p') = \tau$. Obviously, p' is a path from v_s to v_e and it does not travel through v_i, otherwise it contradicts with $\tau < f(\Phi_{s,i} + \Phi_{i,e})$. Then we have $f(p') < f(\Phi_{s,i} + \Phi_{i,e}) \leq f(p)$. □

The vertex-filtering algorithm is shown in Algorithm 3. The algorithm needs to perform verification for every vertex in $\bar{V}$, then the time complexity of the vertex-filtering algorithm is $O(\bar{V})$. $\bar{V}_f$ is the set of vertices that cannot be filtered in the vertex-filtering step. Let $\bar{G}_f(\bar{V}_f, \bar{E}_f)$ be the induced subgraph of $\bar{V}_f$ on $\bar{G}$. By Theorem 3, we only need to compute the optimal path from v_s to v_e on $\bar{G}_f(\bar{V}_f, \bar{E}_f)$.

4.2 Query processing

We discuss the query processing for two cases: (1) score function is linear; and (2) score function is non-linear.

For case (1), every pair of border vertex v_i and entry v_j can be calculated a score according to $\Phi_{i,j}$, and this score can be regarded as a lower bound of distance from one vertex subset to another. In addition, For every $SP_{(i,j);p}$ in Skyline-Path-Inner-Index I^S_p, the minimum score of the skyline path in $SP_{(i,j);p}$ is exactly the shortest distance from an entry v_i to an exit v_j in V_p. By calculating these scores, the partition-based index becomes an index which is similar to G-tree proposed in [32]. The main difference between G-tree and the partition-based index in this case is that G-tree does not pre-compute border-to-border distances between the set of all borders in the graph. The optimal path can be computed in the similar way as G-tree. Note that although our partition-based index also can handle the case of the linear score functions, but it may not be better than the index that elaborately developed for the linear functions, e.g., CH techniques in [8]. Therefore, our partition-based index is more appropriate for the optimal path query under non-linear functions in the multi-cost graphs.

For case (2), the optimal path problem is NP-hard. A *best-first* branch and bound search algorithm can be utilized to compute the optimal path on $\bar{G}_f(\bar{V}_f, \bar{E}_f)$ in the similar way as the algorithm proposed in [31]. Note that $\bar{G}$ is not a simple graph because there are several edges from an entry v_i to an exit v_j in a vertex subset V_p. Given a graph $\bar{G}_f$, a starting vertex

v_s and an ending vertex v_e, all the possible paths started from v_s in $\bar{G}_f$ can be organized in a search tree. Here, the root node represents the starting vertex set $\{v_s\}$. Any non-root node $C = \{v_s, (v_s, v_1), v_1, \cdots, (v_{l-1}, v_l), v_l\}$ represents a path started from v_s. $|C|$ is the number of vertices in C, i.e., $|C| = |\{v|v \in C\}|$. For two different nodes C and C' in the search tree, C is the parent of C' if they satisfy the following two conditions: (i) $C \subset C'$ and $|C'| = |C|+1$; and (ii) $C' \setminus C$ is an *edge-node* set $\{(v_i, v_j), v_j\}$, where v_i and v_j are the ending vertex of path C and C' respectively. In each iteration, a node C is dequeued from the min-heap H. Algorithm extends C by processing the children of C. Assume that the ending vertex of C is v_i. For each edge (v_i, v_j) in $\bar{G}_f$, the algorithm adds the edge-node set $\{(v_i, v_j), v_i\}$ into C to get a child C' of C. Note that there may exist several edges from v_i to v_j when $v_i \in V_p.entry$ and $v_j \in V_p.exit$ and every edge represents a skyline path from v_i to v_j in G_p. The similar pruning strategies in [31] can be used to decide whether C' can be pruned or not. If C' cannot be pruned, it will be inserted into the min-heap H. Algorithm terminates when H is empty or $f(C)$ are not less than the minimum score of the path from v_s to v_e that has been searched for the top element C in H.

The contour skyline set can be used to improve the query efficiency. For an entry v_i and an exit v_j in a cluster V_p, we use $\mathsf{e}_{i,j} = \{(v_i, v_j)^1, \cdots, (v_i, v_j)^m\}$ to denote the multiple edges from v_i to v_j. Each $(v_i, v_j)^\alpha \in \mathsf{e}_{i,j}$ represents a skyline path in $SP_{(i,j);p}$. In each iteration, a node C is to be expanded. Let v_i be the ending vertex of C. If v_i is an entry of a cluster $V_p (V_p \neq V_s$ and $V_p \neq V_e)$, then for each $v_j \in V_p.exit$, we do not need to add every edge-node set $\{(v_i, v_j)^\alpha, v\} (1 \leq \alpha \leq m)$ into C to get a child C' of C. Let $CS_{(i,j);p} = \{cp_1, \cdots, cp_r\}$ be the contour skyline set of $SP_{(i,j);p}$. Each $cp_x \in CS_{(i,j);p}$ corresponds to a group R_x of the skyline paths in $SP_{(i,j);p}$ (recall r-partition), then cp_x corresponds to a group $\mathsf{e}^x_{i,j}$ of edges in $\mathsf{e}_{i,j}$, where $\mathsf{e}^x_{i,j} = \{(v_i, v_j)^{x_1}, \cdots, (v_i, v_j)^{x_t}\}$, $\mathsf{e}^x_{i,j} \subset \mathsf{e}_{i,j}$. Each $(v_i, v_j)^{x_\beta} \in \mathsf{e}^x_{i,j}$ represents a skyline path in R_x. cp_x can be considered as an edge from v_i to v_j and then $\{cp_x, v_j\}$ can be added into C to get a virtual child C' of C. C' corresponds to a children group $C'_x = \{C'_{x_1}, \cdots, C'_{x_t}\}$ of C, where each $C'_{x_\beta} (1 \leq \beta \leq t)$ is a child of C, C'_{x_β} is obtained by adding the edge-node set $\{(v_i, v_j)^{x_\beta}, v_j\}$ into C. Because cp_x is the LBOP of R_x, then cp_x is the LBOP of $\mathsf{e}^x_{i,j}$. Thus, we have $C' \prec C'_{x_\beta}$ for any β, $1 \leq \beta \leq t$. If the virtual node C' can be pruned, then all C'_{x_β} in C'_x can be pruned.

For the time complexity of the best-first branch and bound search algorithm, it is essentially to find the optimal path on graph $\bar{G}_f(\bar{V}_f, \bar{E}_f)$. Let the maximum out-degree of $\bar{G}_f$ be λ, i.e., $\lambda = \max\{d^+(v)|v \in \bar{V}_f\}$. In the worst case, the search space is $O(\lambda^{|\bar{V}_f|})$. The optimal path problem is NP-hard, thus there doesn't exist polynomial time algorithm to find optimal solution for this problem if $P \neq NP$. The experimental results validate the efficiency of the best-first search algorithm with our partition-based index, even though the theoretical time complexity is large.

5 Performance study

In this section, we test the partition-based index on six real-life networks including road networks, social network, etc. The details of these networks are shown in Table 1. All experiments were done on a 3.0 GHz Intel Pentium Core i5 CPU PC with 64GB main memory. All algorithms are implemented by Visual C++.

For each network, we randomly assigned d kinds of cost to every edge ($d \in \{2, 3, 4, 5\}$). Specifically, for the road networks CARN, EURN and WURN, they have two kinds of actual

Table 1 Dataset characteristics

Dataset	Category	Number of vertices	Number of edges
CAITN	IP network	4,837	17,426
EuAll	email network	11,521	32,389
Slashdot	social network	20,639	187,672
HepPh	citation network	34,546	421,578
CARN	road network	21,047	21,692
EURN	road network	3,598,623	8,778,114
WURN	road network	6,262,104	15,248,146

cost, physical distance and transit time. We use the physical distance and transit time as the first two kinds of cost and generate the other $d-2$ kinds of cost. We randomly generate 1,000 pairs of vertices and query the optimal path for every pair . The reported querying time is the average time on each dataset. The score function is $f(w_1, \cdots, w_d) = \sum_{i=1}^{d} w_i^2$.

We compare our method with A* algorithm [16], genetic algorithm(GA) [5] and LEXGO* algorithm [19], which are three the state of the art heuristic algorithms for querying skyline paths over multi-cost graphs. Note that skyline paths essentially are a candidate set for an optimal path query, thus more time is necessary to seek out the optimal path from the skyline paths for these methods. The experimental results present the querying time of skyline path by these heuristic methods are always much larger than the optimal path by our method, even though the time is not counted in for finding an optimal one from all the skyline paths. We also compare our method with BF-Search [31], which uses a naive index to find the optimal path in the multi-cost networks under the non-linear functions.

Exp-1: Querying time As shown in Table 2, we investigate the querying time on five datasets by comparing the partition-based index with A* algorithm, genetic algorithm, LEXGO* algorithm and BF-Search for $d \in \{2, 3, 4, 5\}$. In this experiment, the number of vertex subsets is $k = 50$. For all networks, the querying time of the partition-based index is always in order of magnitude less than the others. The reason is that the partition-based index pre-computes the LBOP, skyline paths and contour skyline for any pair of entry and exit in every vertex subset and a large proportion of the vertices are filtered in the vertex-filtering phase.

Exp-2: Index size The index size is shown in Table 3. We compare the size of the partition-based index with the BF-Search for $d \in \{2, 3, 4, 5\}$. A* algorithm, genetic algorithm and LEXGO* algorithm are not listed here because they do not use index. The number k is also 50. We find the size of the partition-based index is much smaller than BF-Search. These results indicate the partition-based index is space efficient and it is more suitable for the large networks.

Exp-3: Index construction time The index construction time is shown in Table 4. We compare the construction time of the partition-based index with the BF-Search for $d \in \{2, 3, 4, 5\}$. A* algorithm, genetic algorithm and LEXGO* algorithm are not listed here because they do not use index. The number k is also 50. We find the construction time of the partition-based index is always smaller than BF-Search. These results indicate the partition-based index is more time efficient for index constructing.

Table 2 Online querying time in second

Dimension	Dataset	A*	GA	LEXGO*	BF-Search	PB-Index
$d = 2$	CAITN	28.37	8.76	10.13	0.0374	0.0041
	CARN	121.25	36.87	32.71	0.0733	0.0115
	EuAll	211.76	92.28	79.27	0.1471	0.0062
	Slashdot	879.98	193.91	201.36	4.8139	0.0871
	HepPh	1934.52	303.64	288.71	17.653	0.2194
$d = 3$	CAITN	47.26	12.42	16.52	0.0515	0.0071
	CARN	219.38	68.73	79.83	0.0851	0.0189
	EuAll	336.52	155.34	132.46	0.2019	0.0113
	Slashdot	1127.62	316.77	289.71	6.2506	0.1027
	HepPh	3253.43	589.32	573.13	21.467	0.2938
$d = 4$	CAITN	67.38	22.61	25.57	0.0917	0.0113
	CARN	389.72	113.32	135.78	0.1324	0.0242
	EuAll	557.13	289.44	277.31	0.3745	0.0213
	Slashdot	1979.52	553.68	497.23	9.8705	0.1832
	HepPh	4791.72	783.41	747.91	37.833	0.5013
$d = 5$	CAITN	88.42	37.98	38.31	0.1415	0.0188
	CARN	512.75	189.67	197.93	0.2031	0.0403
	EuAll	787.39	479.73	451.82	0.5719	0.0374
	Slashdot	3241.85	898.63	831.42	15.312	0.3146
	HepPh	7127.05	1124.71	997.34	59.633	0.9708

Table 3 Index size in MB

	$d = 2$		$d = 3$		$d = 4$		$d = 5$	
Dataset	BFS	PBIndex	BFS	PBIndex	BFS	PBIndex	BFS	PBIndex
CAITN	115.99	6.21	203.78	13.52	269.82	16.37	317.41	19.86
CARN	2600.68	93.85	4398.95	163.98	5082.71	197.58	5982.95	232.65
EuAll	796.33	20.83	1333.86	39.23	1617.85	50.16	2041.43	57.88
Slashdot	1746.39	47.21	3136.24	81.75	3849.72	96.97	4340.19	114.61
HepPh	4124.96	138.74	6460.35	224.02	7171.62	279.83	7895.45	332.65

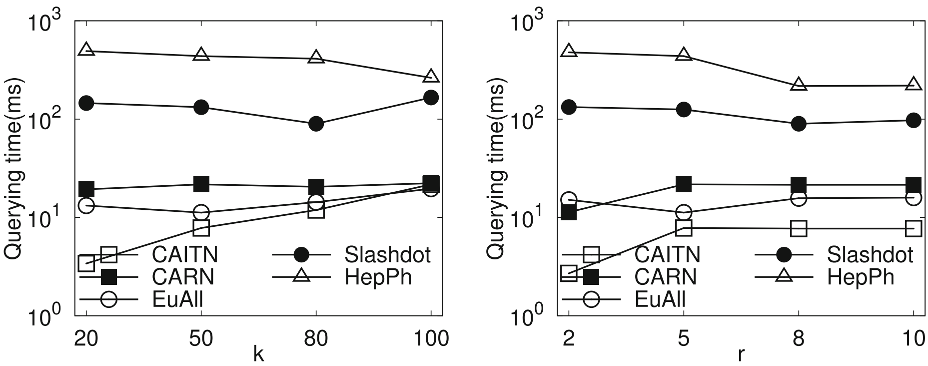
Table 4 Index construction time in second

	$d = 2$		$d = 3$		$d = 4$		$d = 5$	
Dataset	BFS	PBIndex	BFS	PBIndex	BFS	PBIndex	BFS	PBIndex
CAITN	127.57	102.45	196.33	121.87	234.74	159.69	297.85	193.72
CARN	573.96	495.76	878.82	612.51	1192.43	832.01	1679.31	984.65
EuAll	351.69	312.69	468.27	427.81	595.77	531.36	732.85	694.79
Slashdot	662.41	521.39	987.34	671.47	1482.65	987.48	1968.19	1124.58
HepPh	912.43	873.91	1317.46	1035.11	1874.39	1621.74	2679.91	2247.35

Exp-4: Impact of vertex-filtering We investigate the effectiveness of the vertex-filtering algorithm in Table 5. In this experiment, $k = 50$ and $d = 2$. From Table 5, we find the vertex-filtering algorithm can filter at least 50% vertices for each dataset. We find $|\bar{E}|$ may be larger than $|E|$, where $|\bar{E}|$ and $|E|$ are the number of vertices in the shrunk graph $\bar{G}$ and the original graph G respectively. It is because that there are multiple edges between every pair of entry v_i and exit v_j in each V_p ($V_p \neq V_s$ and $V_p \neq V_e$) in $\bar{G}$. $Avg.|SP_{(i,j);p}|$ in Table 5 is the average number of the edges between any pair of entry v_i and exit v_j in the same vertex subset. In fact, for each pair of entry v_i and exit v_j, $|SP_{(i,j);p}| \ll |P_{(i,j);x}|$, where $|P_{(i,j);x}|$ is the number of all the possible paths from i to j in G_x. Therefore, even though $|\bar{E}| > |E|$, our algorithm on $\bar{G}$ is more efficient than that on G because many paths from an entry to an exit have been filtered by $SP_{(i,j);p}$. In addition, each edge $(v_i, v_j)^\alpha$ from an entry v_i to an exit v_j in $\bar{G}$ represents a skyline path from v_i to v_j. When the algorithm expands a node C whose ending vertex is v_i, C's children in $\bar{G}$ are more possible to be pruned than that in G.

Exp-5: Impact of *k* and *r* We investigate the impact of the number k of the vertex subsets and the size r of the contour skyline set. The experimental results are shown in Figure 4. For k, an appropriate k makes the number of the entries and the exits smaller in $\bar{G}$ and thus the querying time is less. A larger or smaller k will increase the querying time. In Figure 4a, we find the optimal k are distinct for the different datasets. For example, the optimal k is 50 for Euall dataset but it is 80 for Slashdot dataset. For r, the skyline points in a group are more proximity under a larger r and then the algorithm is more effective to prune a virtual node C' as the discussion in Section 4.2. On the other hand, a larger r results in the more contour skyline points and then the querying time increases. In two extreme cases, when $r = 1$, the only contour skyline point is the LBOP of $SP_{(i,j);p}$, and when $r = |SP_{(i,j);p}|$, the contour skyline set is exactly $SP_{(i,j);p}$. For these two cases, the contour skyline set cannot work well. We find the optimal r is also distinct for the different datasets. The optimal r is 5 for EuAll dataset and it is 8 for Slashdot and HepPh datasets.

Exp-6. Scalability We evaluate the scalability of our method in Figure 5. We investigate the querying time by varying the number of vertices from one million to three millions on EURN and WURN dataset for $d = 2$ and $d = 3$. For each graph, $k = 10^{-3}n$, where n is the number of vertices in graph. We compare our method with BF-Search, GA algorithm and LEXGO* algorithm. The experimental results show our method is always in order of



(a) impact of k (b) impact of r

Figure 4 Impact of k and r

Table 5 Impact of vertex-filtering

Dataset	$\lvert\bar{V}\rvert$	$\lvert\bar{E}\rvert$	$\lvert\bar{V}_f\rvert$	$\lvert\bar{E}_f\rvert$	$Avg.\lvert SP_{(i,j);x}\rvert$
CAITN	746	19,132	368	9,560	11.17
CARN	1,268	27,338	539	12,057	6.02
Enron	1,073	29,418	471	13,715	14.78
Slashdot	1,782	293,877	936	198,429	43.16
HepPh	3,832	1,718,753	1,297	646,396	55.31

magnitude faster than others and it can perform efficiently even though the number of vertices is larger than three million. The index construction time and index size are reported in Figures 6 and 7 respectively. A* algorithm, genetic algorithm and LEXGO* algorithm are not reported in Figures 6 and 7 because they do not use index. The experimental results show our method is always better than BF-Search. It indicates our method are also suitable for large multi-cost graphs.

Exp-7. Impact of score function As shown in Figure 8, we investigate the impact of score function and report the querying time by varying the number of vertices from one million to three millions on EURN and WURN dataset for $d = 2$ and $d = 3$. For each graph,

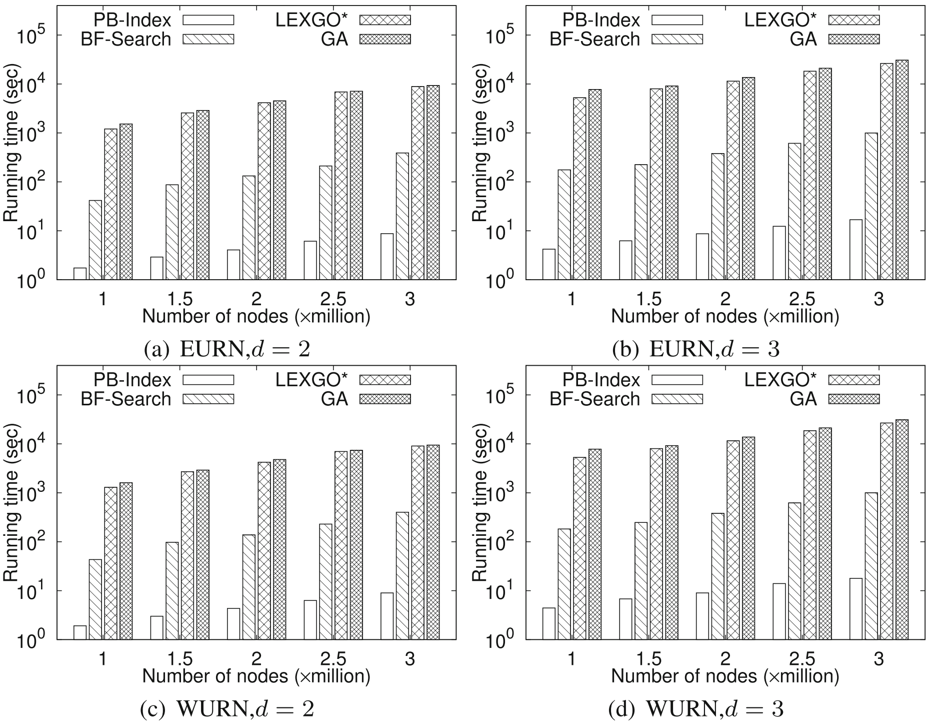
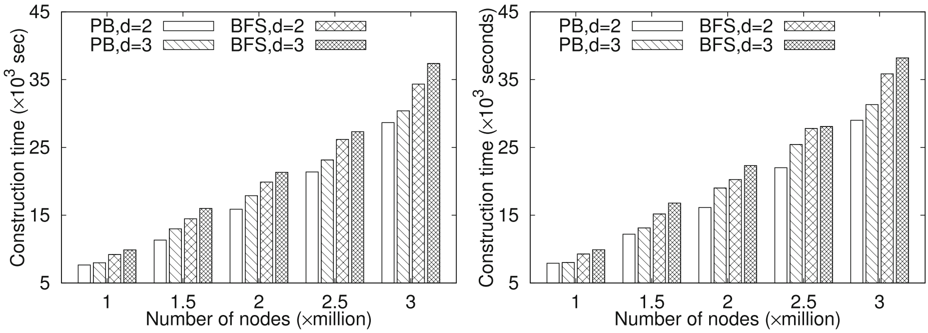


Figure 5 Querying time to large graphs

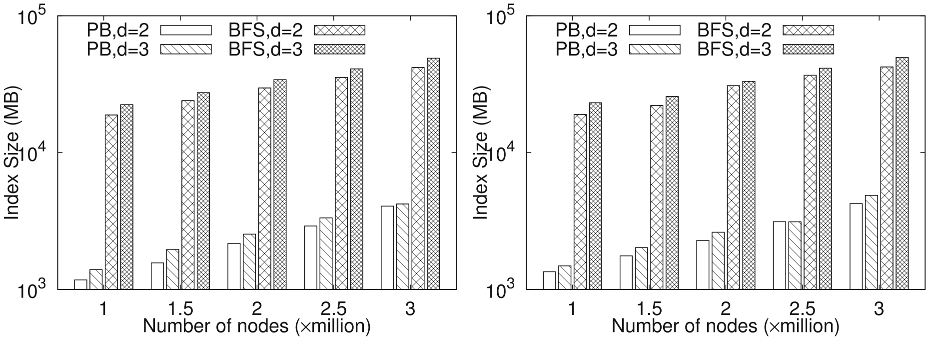


(a) EURN (b) WURN

Figure 6 Index construction time for large graphs

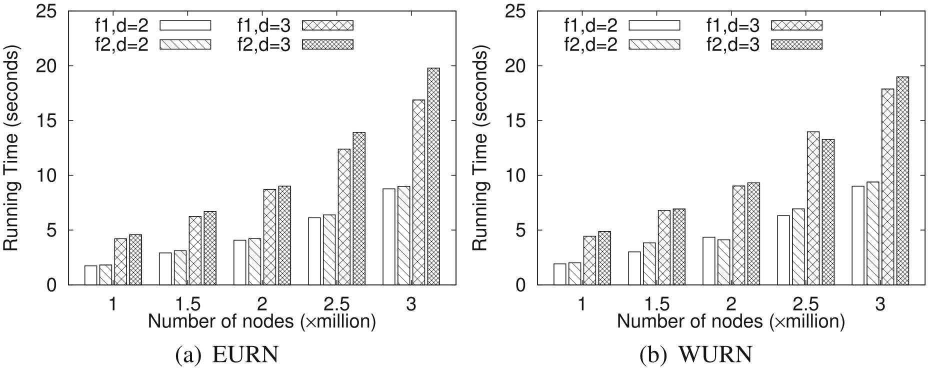
$k = 10^{-3}n$, where n is the number of vertices in graph. We compare two score functions, $f_1(w_1, \cdots, w_d) = \sum_{i=1}^{d} w_i^2$ and $f_2(w_1, \cdots, w_d) = \sum_{i=1}^{d} w_i^{d-i+1}$. The experimental results show the querying time does not change significantly for different score functions and it indicates our method are not sensitive to the score function.

Exp-8. Querying time for the linear score functions As shown in Figure 9, we investigate the querying time for the linear functions by varying the number of vertices from one million to three millions on EURN and WURN datasets. We compare our method with the CH techniques proposed in [8] and Dijkstra algorithm for the linear functions $f_1(w_1, \cdots, w_d) = \sum_{i=1}^{d} w_i$ when $d = 2$ and $d = 3$. The experimenal results show both our method and CH techniques are much more efficient than Dijkstra algorithm but the querying time of our method is slghtly larger than CH techniques. The reason is that the CH techniques is developed for the linear functions by computing hyperplane to make the query more efficient. However, CH techniques cannot handle the case of non-linear function because it is impossible to calculate hyperplane for the arbitrary non-linear functions. The experimental results shows our partition-based index is more appropriate for the optimal path query under non-linear functions in the multi-cost graphs.



(a) EURN (b) WURN

Figure 7 Index size for large graphs

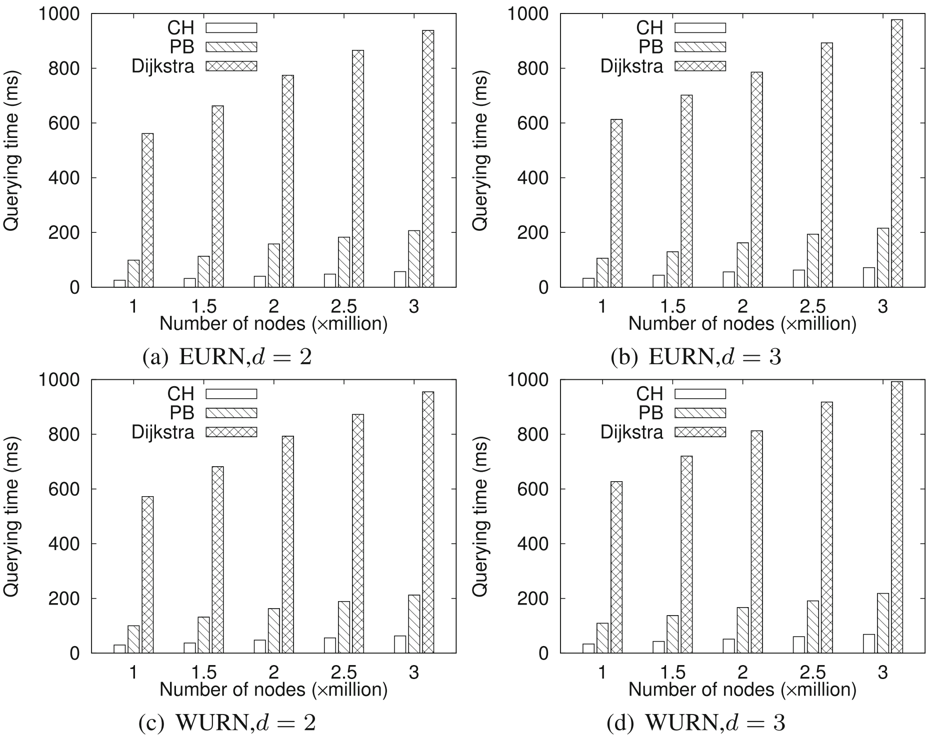


(a) EURN (b) WURN

Figure 8 Impact of score function

6 Related work

The existing works for the shortest path problem propose various index techniques to enhance the efficiency of the shortest path query for large graphs. *The shortest path quad tree* scheme is proposed in [24], which pre-computes the shortest paths for every two vertices in a graph and organizes them by a quad tree. This method is not applicable to the



(a) EURN,$d = 2$ (b) EURN,$d = 3$

(c) WURN,$d = 2$ (d) WURN,$d = 3$

Figure 9 Querying time for linear score functions optimal path problem in the multi-cost graphs. Because the score functions given by different users may be different, the quad tree constructed according to one score function cannot answer the optimal path query under the other functions. Xiao et al. in [28] proposes the concept of the compact BFS-trees where the BFS-trees are compressed by exploiting the symmetry property of the graphs. Wei et al. in [27] proposes a novel method named TEDI, which utilizes the tree decomposition theory to build an index and process the shortest path query. Cheng et al. in [4] proposes a disk-based index for the single-source shortest path or distance queries. This index is a tree-structured index constructed based on the concept of vertex cover and it is I/O-efficient when the input graph is too large to fit in main memory. Rice et al. in [22] introduces a new shortest path query type in which dynamic constraints may be placed on the allowable set of edges that can appear on a valid shortest path. They formalize this problem as a specific variant of formal language constrained shortest path problems and then they propose the generalized shortest path queries in the following work [23]. Zhu et al. in [33] presents AH index to narrow the gap between theory and practice. Landmark-based techniques have been widely used to estimate the distance between two vertices in a graph in many applications [2, 10, 20]. Goldberg et al. in [10] choose some anchor vertices called landmark and pre-computes for each vertex its graph distance to all anchor vertices. A distance vector is created from these distances. A lower bound derived from the distance vector can be used by A^* algorithm to guide the shortest path search. Qiao et al. in [21] propose a query-dependent local landmark scheme, which identifies a local landmark close to the specific query nodes and provides a more accurate distance estimation than the traditional global landmark approaches. The latest work [2] proposes a new exact method based on *distance-aware 2-hop cover* for the distance queries. All the above methods utilize the following property in the shortest path: any sub-path of a shortest path is also a shortest path. Therefore, they only need to maintain the shortest paths among the vertices in the index and compute the shortest path by concatenating the sub shortest paths in the index. However, in the multi-cost graphs, this property does not hold. Therefore, these methods cannot solve the optimal path problem in the multi-cost graphs.

In recent years, several works [5, 6, 11, 15–17, 19, 25, 30] study the multi-criteria shortest path (MCSP) problem on multi-cost graphs. Given a starting vertex s and an ending vertex t, it is to find all the skyline paths from s to t. Most existing works on MCSP are heuristic algorithms based on the following property: any sub-path of a skyline path is also a skyline path. To compute a skyline path p, these methods need to expand all the skyline paths from the starting vertex to a vertex v for every $v \in p$. In [30], time varying uncertainty is integrated into the multi-cost graph model and the optimal route is defined as a skyline path under stochastic dominance. The linear skyline path problem is studied in [25], it is to the paths which are the optimal under the weighted sum or linear combination. In [11], the skyline query is to find a set of constrained objects not dominated by any other one under considering walk distance, object attributes and path cost. However, our problem is to find an optimal path from a starting vertex to an end vertex under a given score function. The difference between MCSP and our problem is as follows. MCSP is to find all skyline paths but our problem is only to find one path that is the most optimal under the score function. It is obvious that skyline paths are a candidate set of the optimal path. However, the time cost is too expensive to find an optimal path by exhausting all skyline paths. Moreover, these works do not develop any index technique to facilitate the skyline path querying. Mouratidis et al. in [18] studies the skyline queries and the top-k queries on the multi-cost transportation networks. For any vertex v in graph, all the distances on the different dimensions between v and the query point form the cost vector of v. The definition of the cost vector in this work is different from ours and the query results are points but not paths. Therefore, the methods in this work cannot be applied to the optimal path problem in this paper. There are also some works [8] about the optimal path problem under linear score funtions and propose some query optimization techniques, e.g., as Contraction Hierarchies. The optimal path problem under linear score funtions is essentially equivalent to the traditional shortest path problem and thus it can be solved by existing algorithms, e.g., Dijkstra algorithm. The methods for the optimal path problem under linear score funtions are also based on main idea of Dijkstra algorithm. However, the optimal path problem is NP-hard under the non-linear functions and then these methods cannot be used to solve this problem under arbitrary score function.

7 Conclusion

In this paper, we study the problem of finding the optimal path in the multi-cost networks. We prove this problem is NP-hard and propose a novel partition-based index with contour skyline techniques. We also propose a vertex-filtering algorithm to facilitate the query processing. We conduct extensive experiments and the experimental results validate the efficiency of our method.

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