3 The Church-Turing Thesis

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2016

Outline

1. Turing Machines
2. Variants of Turing Machines
3. The Definition of Algorithm

Outline

1 Turing Machines
- Formal Definition of a Turing Machine
- Examples of Turing Machines

2 Variants of Turing Machines

3 The Definition of Algorithm

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Even a Turing machine cannot solve certain problems

- these problems are beyond the theoretical limits of computation

艾伦·图灵（Alan Turing）

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June 23, 1912 – June 7, 1954 (aged 41)

英国数学家、逻辑学家

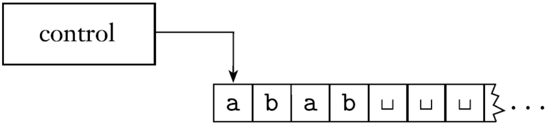
“计算机科学之父”

“人工智能之父”

Turing Machines

The Turing machine model uses an infinite tape as its unlimited memory.

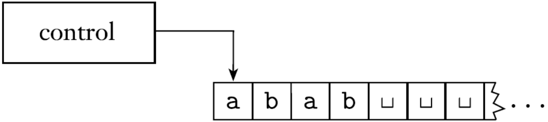
- It has a tape head that can read and write symbols and move around on the tape.
- Initially the tape contains only the input string and is blank everywhere else.
- If the machine needs to store information, it may write this information on the tape.



Turing Machines

The Turing machine model uses an infinite tape as its unlimited memory.

- To read the information that it has written, the machine can move its head back over it.
- The machine continues computing until it decides to produce an output.
- The outputs ***accept*** and ***reject*** are obtained by entering designated accepting and rejecting states.
- If it doesn't enter an accepting or a rejecting state, it will go on forever, never ***halting***.



Turing Machines

The differences between finite automata and Turing machines

1. A Turing machine can both write on the tape and read from it.
2. The read–write head can move both to the left and to the right.
3. The tape is infinite.
4. The special states for rejecting and accepting take effect immediately.

Turing Machines

Example (Turing machine M_1)

Let's introduce a Turing machine M_1 for testing membership in the language

$$B = \{w\#w \mid w \in \{0,1\}^*\}$$

We want M_1 to accept if its input is a member of B and to reject otherwise.

Turing Machines

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On input string w:

1. Zig-zag across the tape to corresponding positions on either side of the $\#$ symbol to check whether these positions contain the same symbol. If they do not, or if no $\#$ is found, reject. Cross off symbols as they are checked to keep track of which symbols correspond.

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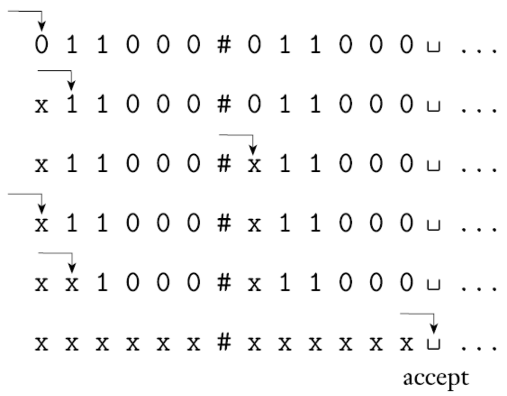
2. When all symbols to the left of the $\#$ have been crossed off, check for any remaining symbols to the right of the $\#$. If any symbols remain, reject; otherwise, accept.

Turing Machines

Example (Turing machine M_1)

several nonconsecutive snapshots of M_1's tape after it is started on input $011000\#011000$.



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6. $q_{\text{accept}} \in Q$ is the ***accept state***, and
7. $q_{\text{reject}} \in Q$ is the ***reject state***, where $q_{\text{accept}} \neq q_{\text{reject}}$.

Configurations of a Turing Machine

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- the current state
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- the current head location

A configuration of the Turing machine

uqv

- the current state is q
- the current tape contents is uv
- the current head location is the first symbol of v

The tape contains only blanks following the last symbol of v.

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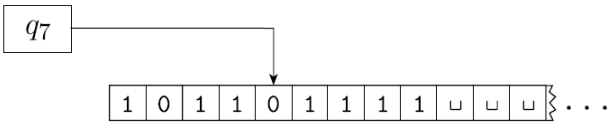

How does a Turing machine compute

Configuration C_1 ***yields* 产生** configuration C_2

- if the Turing machine can legally go from C_1 to C_2 in a single step.

Formalization

Suppose that $a, b \in \Gamma$, $u, v \in \Gamma^*$, $q_i, q_j \in Q$

- uaq_ibv yields uq_jacv

if $\delta(q_i, b) = (q_j, c, \mathsf{L})$

- uaq_ibv yields $uacq_jv$ if $\delta(q_i, b) = (q_j, c, \mathsf{R})$

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Formalization

Special cases occur when the head is at one of the ends.

- For the left-hand end
  - left-moving: $q_i bv$ yields $q_j cv$
  - right-moving: $q_i bv$ yields $cq_j v$
- For the right-hand end
  - uaq_i is equivalent to $uaq_i\sqcup$

How does a Turing machine compute

TM M on input w

- ***start configuration***: $q_0 w$
- ***accepting configuration***: $\cdots q_{\text{accept}} \cdots$
- ***rejecting configuration***: $\cdots q_{\text{reject}} \cdots$

Accepting and rejecting configurations are ***halting configurations*** and do not yield further configurations.

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Because the machine is defined to halt when in q_{accept} and q_{reject}, we equivalently could have defined the transition function to have the more complicated form

- $\delta : Q' \times \Gamma \to Q \times \Gamma \times \{\mathsf{L}, \mathsf{R}\}$, where $Q' = Q - \{q_{\text{accept}}, q_{\text{reject}}\}$

How does a Turing machine compute

A Turing machine M ***accepts*** input w if a sequence of configurations $C_1, C_2, \ldots, C_k$ exists, where

- C_1 is the start configuration of M on input w,
- each C_i yields C_{i+1}, and
- C_k is an accepting configuration.

The collection of strings that M accepts is ***the language of*** M, or ***the language recognized by*** M, denoted $L(M)$.

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Call a language ***Turing-recognizable*** 图灵可识别的 if some Turing machine recognizes it.

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(It is called a ***recursively enumerable language*** **递归可枚举语言**.)

When we start a Turing machine on an input, three outcomes are possible.

1. accept
2. reject
3. loop

By ***loop*** we mean that the machine simply does not halt.

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- Every decidable language is Turing-recognizable.

Relationship among the classes of languages

Theorem

Every context-free language is decidable.

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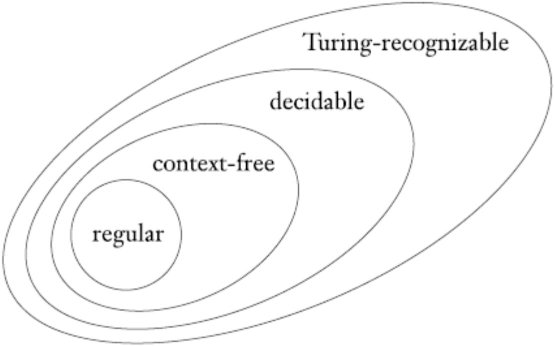

Example (TM M_2 that decides $A = \{0^{2^n} | n \geq 0\}$, Input: 0000)

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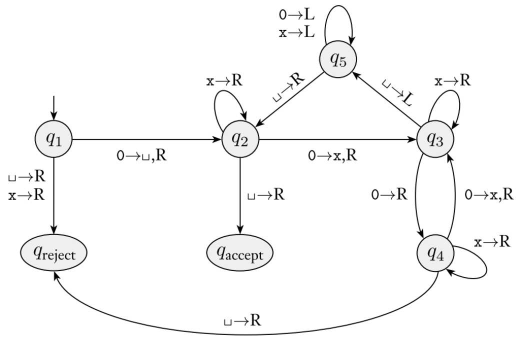

$q_1 0000$
$\sqcup q_2 000$
$\sqcup \mathrm{x} q_3 00$
$\sqcup \mathrm{x} 0 q_4 0$
$\sqcup \mathrm{x} 0 \mathrm{x} q_3 \sqcup$
$\sqcup \mathrm{x} 0 q_5 \mathrm{x} \sqcup$
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$\sqcup \mathrm{x} q_2 \mathrm{xx} \sqcup$
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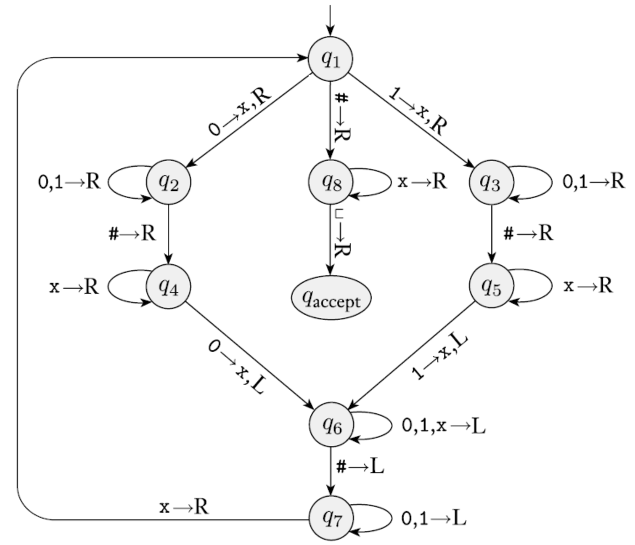
$\sqcup \mathrm{xxx} q_2 \sqcup$
$\sqcup \mathrm{xxx} \sqcup q_{accept}$

Example (TM M_1)



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2 Variants of Turing Machines

- Multitape Turing Machines
- Nondeterministic Turing Machines 非确定型图灵机
- Enumerators
- Equivalence With Other Models

3 The Definition of Algorithm

Multitape Turing Machines 多带图灵机

多带图灵机

- A *multitape Turing machine* is like an ordinary Turing machine with several tapes. Each tape has its own head for reading and writing. Initially the input appears on tape 1, and the others start out blank.

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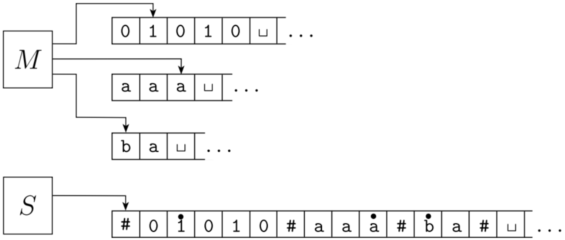

Nondeterministic Turing Machines

- $\delta : Q \times \Gamma \to \mathcal{P}(Q \times \Gamma \times \{\mathsf{L}, \mathsf{R}\})$

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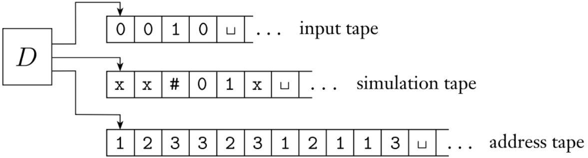

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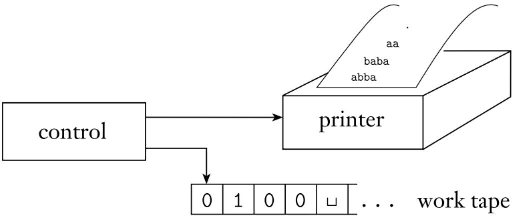

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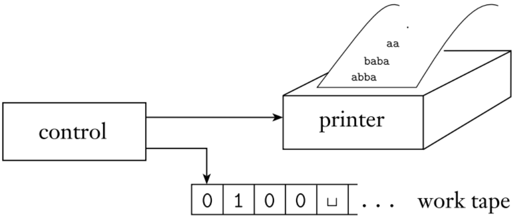

Theorem

A language is Turing-recognizable if and only if some enumerator enumerates it.

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2. Variants of Turing Machines

3. The Definition of Algorithm
   - Hilbert's Problems

Church–Turing thesis

Intuitive notion of algorithms	equals	*Turing machine algorithms*

The Church–Turing Thesis 邱奇-图灵论题