2 Context-Free Languages

(Part 2 of 2)

Yajun Yang

yjyang@tju.edu.cn

School of Computer Science and Technology

Tianjin University

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Outline

Context-Free Languages

regular languages 正则语言

- finite automata: DFA / NFA
- regular expressions

some simple languages, such as $\{0^n1^n \mid n \geq 0\}$, are ***not*** regular languages.

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context-free languages 上下文无关语言

- pushdown automata 下推自动机

- first used in the study of human languages
- in the specification and compilation of programming languages
  - ***parser***
  - the construction of a parser from a context-free grammar

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Pushdown Automata (PDA): we introduce a new type of computational model.

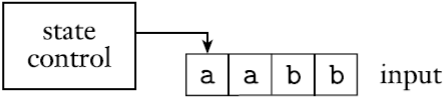
- like NFA but have an extra component called a ***stack***.
- the stack provides additional memory beyond the finite amount available in the control.
- the stack allows PDA to recognize some nonregular languages.

PDA are equivalent in power to CFG

- two options for proving that a language is context free
  - give either a CFG generating it (generator)
  - or a PDA recognizing it (recognizer)

Pushdown Automata 下推自动机

DFA/NFA vs. PDA



Schematic of DFA/NFA

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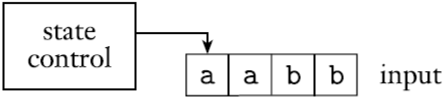

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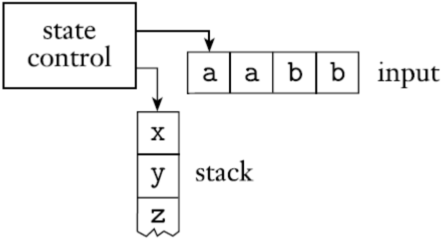

Schematic of PDA

Pushdown Automata 下推自动机

PDA can write symbols on the stack and read them back later.

Writing a symbol ***"pushes down"*** all the other symbols on the stack.

- all access to the stack may be done only at the top: "last in, first out"
- ***pushing***: writing a symbol on the top of the stack
- ***popping***: removing a symbol on the top of the stack

A stack can hold an unlimited amount of information.

the language $\{0^n 1^n \mid n \geq 0\}$

- a DFA/NFA is unable to recognize it.
- A PDA is able to recognize it.

Pushdown Automata 下推自动机

Deterministic and nondeterministic PDA are ***not*** equivalent in power.

- Nondeterministic PDA recognize certain languages that ***no*** deterministic PDA can recognize.
- Recall that DFA and NFA do recognize the same class of languages.
- So the pushdown automata situation is different.
- We focus on nondeterministic PDA because these automata are equivalent in power to CFG.

Formal Definition of a Pushdown Automaton

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6. $F \subseteq Q$ is the set of ***accept states***.

Formal Definition of Computation for a PDA

A PDA $M = (Q, \Sigma, \Gamma, \delta, q_0, F)$ computes as follows.

- It accepts input w if w can be written as $w = w_1 w_2 \cdots w_m$, where $w_i \in \Sigma_\varepsilon$ and

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  2. For $i = 0, \ldots, m-1$, we have $(r_{i+1}, b) \in \delta(r_i, w_{i+1}, a)$, where $s_i = at$ and $s_{i+1} = bt$ for some $a, b \in \Gamma_\varepsilon$ and $t \in \Gamma^*$

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3. $r_m \in F$

Examples of Pushdown Automata

Example (PDA M_1 recognizes the language $\{0^n1^n \mid n \geq 0\}$)

Let M_1 be $(Q, \Sigma, \Gamma, \delta, q_1, F)$, where

$Q = \{q_1, q_2, q_3, q_4\}$

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Input:	0			1			ε		
Stack:	0	\$	ε	0	\$	ε	0	\$	ε
q_1									$\{(q_2, \$)\}$
q_2			$\{(q_2, 0)\}$	$\{(q_3, \varepsilon)\}$					
q_3				$\{(q_3, \varepsilon)\}$				$\{(q_4, \varepsilon)\}$	
q_4									

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State diagram for the PDA M_1

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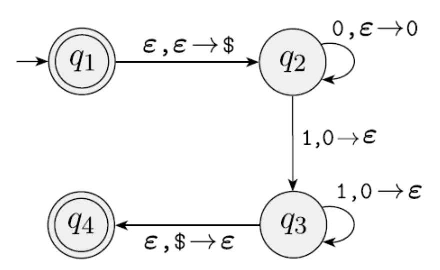

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A pushdown automaton that recognizes the language

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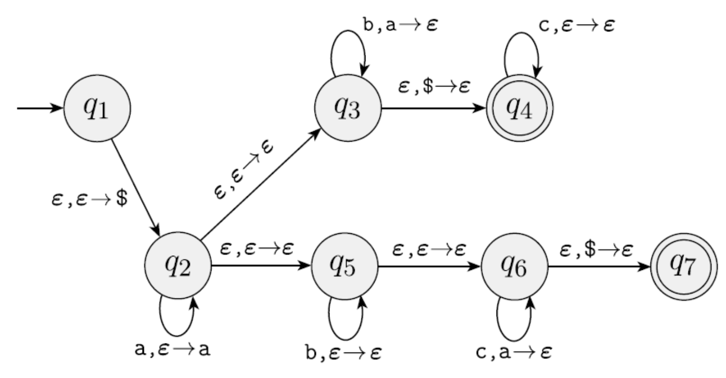

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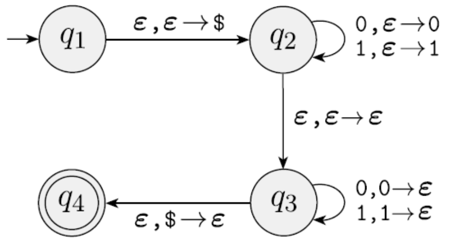

Equivalence With Context-Free Grammars

Theorem

A language is context free if and only if some pushdown automaton recognizes it.

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If a pushdown automaton recognizes some language, then it is context free.

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- Let A be a CFL generated by a CFG G. We convert G into an equivalent PDA P.
- P accepts a input w, if G generates w by a sequence of derivations.
- PDA P begins by writing the start variable on its stack. It goes through a series of intermediate strings. Eventually it may arrive at a string that contains only terminal symbols. Then P accepts if this string is identical to the string it has received as input.

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   3. If the top of stack is the symbol $\$$, enter the accept state. Doing so accepts the input if it has all been read.

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- Let q and r be states of the PDA and let a be in Σ_ϵ and s be in Γ_ϵ. P goes from q to r when it reads a and pops s.

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- Let q and r be states of the PDA and let a be in Σ_ϵ and s be in Γ_ϵ. P goes from q to r when it reads a and pops s.
- Push the entire string $u = u_1, \cdots, u_l$ on the stack at the same time.

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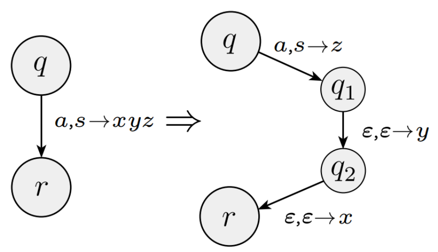

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- The states of P are $Q = \{q_{\text{start}}, q_{\text{loop}}, q_{\text{accept}}\} \cup E$, E is s the set of states we need for implementing the shorthand just described.

$\square$

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  - $\delta(q_{\text{loop}}, \epsilon, \$) = \{(q_{\text{accept}}, \epsilon)\}$, the empty stack marker $\$$ is on the top of the stack.

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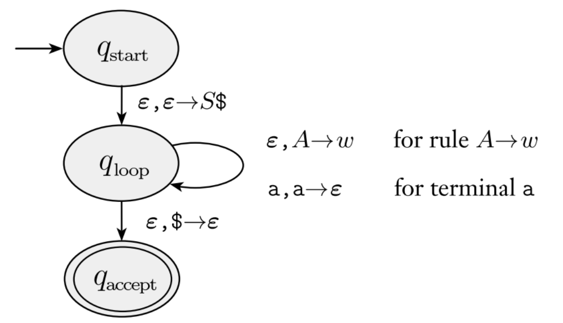
Proof.

The state diagram is shown in the following figure



Equivalence With Context-Free Grammars

Example (construct a PDA P from the following CFG G.)

$S \rightarrow aTb|b$, $T \rightarrow Ta|\epsilon$

Equivalence With Context-Free Grammars

Example (construct a PDA P from the following CFG G.)

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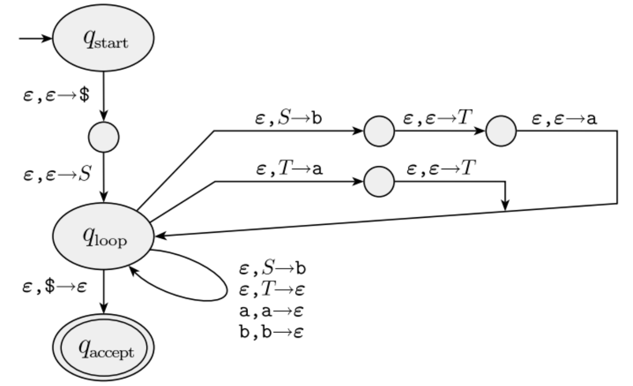
The transition function is shown in the following diagram



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- Given a PDA P, make a CFG G generating all the strings that P accepts.
- P accepts a input w, if G generates w by a sequence of derivations.
- Design a grammar that does somewhat more. For each pair of states p and q in P, the grammar will have a variable A_{pq}. This variable generates all the strings that can take P from p with an empty stack to q with an empty stack.

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- Giving P features 1 and 2 is easy.

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  - we replace each transition that neither pops nor pushes with a two transition sequence that pushes then pops an arbitrary stack symbol.

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- Two possibilities occur during P's computation on x. Either the symbol popped at the end is the symbol that was pushed at the beginning, or not.

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Proof.

Given $P = (Q, \Sigma, \Gamma, \delta, q_0, \{q_{\text{accept}}\})$, construct G. The variables of G are $\{A_{pq} | p, q \in Q\}$. The start variable is $A_{q_0, q_{\text{accept}}}$. We describe G's rules in three parts.

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- For each $p, q, r, s \in Q$, $u \in \Gamma$, and $a, b \in \Sigma_\epsilon$, if $\delta p, a, \epsilon$ contains (r, u) and $\delta(s, b, u)$ contains (q, ϵ) put the rule $A_{pq} \rightarrow aA_{rs}b$ in G.

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- For each $p, q, r \in Q$, put the rule $A_{pq} \to A_{pr}A_{rq}$ in G.

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- For each $p, q, r, s \in Q$, $u \in \Gamma$, and $a, b \in \Sigma_\epsilon$, if $\delta p, a, \epsilon$ contains (r, u) and $\delta(s, b, u)$ contains (q, ϵ) put the rule $A_{pq} \to aA_{rs}b$ in G.
- For each $p, q, r \in Q$, put the rule $A_{pq} \to A_{pr}A_{rq}$ in G.
- Finally, for each $p \in Q$, put the rule $A_{pp} \to \epsilon$ in G.

$\square$

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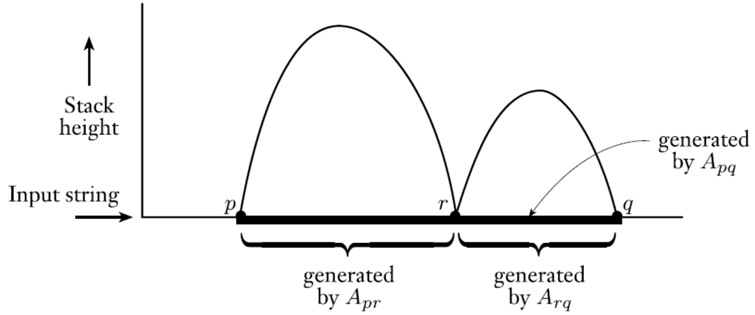

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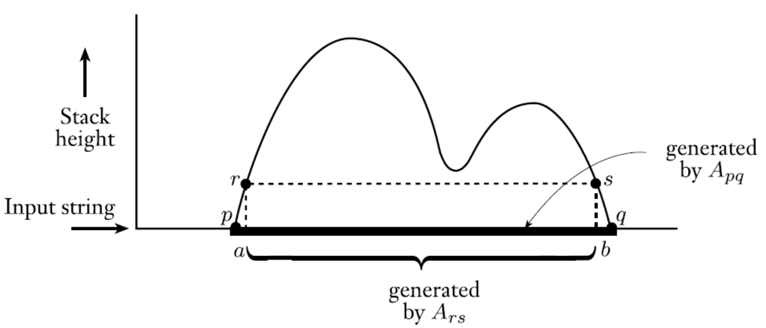

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If A_{pq} generates x, then x can bring P from p with empty stack to q with empty stack.

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If x can bring P from p with empty stack to q with empty stack, A_{pq} generates x.

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We prove this claim by induction on the number of steps in the derivation of x from A_{pq}.

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We prove this claim by induction on the number of steps in the derivation of x from A_{pq}.

- ***Basis***: The derivation has 1 step. A derivation with a single step must use a rule whose right-hand side contains no variables. The only rules in G where no variables occur on the right-hand side are $A_{pp} \to \epsilon$.

□

Equivalence With Context-Free Grammars

Proof.

Induction step: Assume true for derivations of length at most k, where $k \geq 1$, and prove true for derivations of length $k+1$. Suppose $A_{pq} \overset{*}{\Rightarrow} x$ with $k+1$ steps. The first step in this derivation is either $A_{pq} \Rightarrow aA_{rs}b$ or $A_{pq} \Rightarrow A_{pr}A_{rq}$. We handle these two cases separately.

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In the first case, consider the portion y of x that A_{rs} generates, $x = ayb$.

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In the first case, consider the portion y of x that A_{rs} generates, $x = ayb$.

- $A_{rs} \overset{*}{\Rightarrow} y$ with k steps, then P can go from r on empty stack to s on empty stack.

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In the first case, consider the portion y of x that A_{rs} generates, $x = ayb$.

- $A_{rs} \overset{*}{\Rightarrow} y$ with k steps, then P can go from r on empty stack to s on empty stack.
- Because $A_{pq} \to aA_{rs}b$ is a rule in G, $\delta(p, a, \epsilon)$ contains (r, u) and $\delta(s, b, u)$ contains (q, ϵ), for some stack symbol u.

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- $A_{rs} \overset{*}{\Rightarrow} y$ with k steps, then P can go from r on empty stack to s on empty stack.
- Because $A_{pq} \to aA_{rs}b$ is a rule in G, $\delta(p, a, \epsilon)$ contains (r, u) and $\delta(s, b, u)$ contains (q, ϵ), for some stack symbol u.
- x can bring it from p with empty stack to q with empty stack.

□

Equivalence With Context-Free Grammars

Claim

If A_{pq} generates x, then x can bring P from p with empty stack to q with empty stack.

Proof.

In the second case, consider the portions y and z of x that A_{pr} and A_{rq} respectively generate, $x = yz$.

- $A_{pr} \overset{*}{\Rightarrow} y$ in at most k steps and $A_{rq} \overset{*}{\Rightarrow} z$ in at most k steps.

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- $A_{pr} \overset{*}{\Rightarrow} y$ in at most k steps and $A_{rq} \overset{*}{\Rightarrow} z$ in at most k steps.
- y can bring P from p to r, and z can bring P from r to q, with empty stacks at the beginning and end.

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This completes the induction step. □

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If a computation has 0 steps, it starts and ends at the same p. We show that $A_{pp} \overset{*}{\Rightarrow} x$. In 0 steps, P cannot read any characters, so $x = \epsilon$. By construction, G has the rule $A_{pp} \rightarrow \epsilon$, so the basis is proved. □

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If x can bring P from p with empty stack to q with empty stack, A_{pq} generates x.

Proof.

Induction step: Assume true for computations of length at most k, where $k \geq 0$, and prove true for computations of length $k+1$.

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If x can bring P from p with empty stack to q with empty stack, A_{pq} generates x.

Proof.

Induction step: Assume true for computations of length at most k, where $k \geq 0$, and prove true for computations of length $k+1$.

Suppose that P has a computation wherein x brings p to q with empty stacks in $k+1$ steps. Either the stack is empty only at the beginning and end of this computation, or it becomes empty elsewhere, too.

□

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- Consider $x = ayb$, P can go from r with an empty stack to s with an empty stack on input y, the computation on y has $(k+1) - 2 = k - 1$ steps.

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- $A_{rs} \overset{*}{\Rightarrow} y$ and hence $A_{pq} \overset{*}{\Rightarrow} x$

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- We have $A_{pr} \overset{*}{\Rightarrow} y$ and $A_{rq} \overset{*}{\Rightarrow} z$. Because $A_{pq} \rightarrow A_{pr}A_{rq}$ in G, then $A_{pq} \overset{*}{\Rightarrow} x$.

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Proof is complete. □

Equivalence With Context-Free Grammars

Corollary

Every regular language is context free.

Equivalence With Context-Free Grammars

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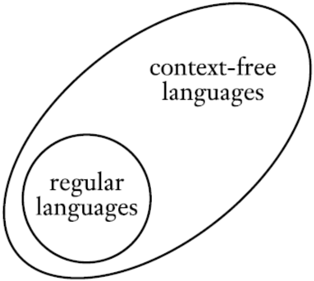

Outline

1. Context-Free Grammars
2. Pushdown Automata
3. Non-Context-Free Languages
   - The Pumping Lemma for Context-Free Languages
4. Deterministic Context-Free Languages

The Pumping Lemma for Context-Free Languages

Theorem (Pumping lemma for context-free languages 泵引理)

If A is a context-free language, then there is a number p (the pumping length) where, if s is any string in A of length at least p, then s may be divided into five pieces, $s = uvxyz$, satisfying the conditions:

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2. *$|vy| > 0$, and*
3. *$|vxy| \leq p$.*

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- On this long path, some variable symbol R must repeat because of the pigeonhole principle.

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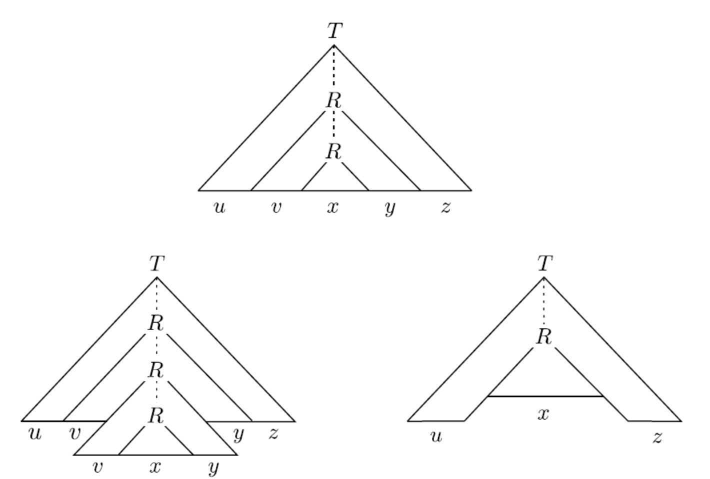
- Let s be a very long string in A, it is derivable from G and has a parse tree. The parse tree must contain a long path from the root to one of a leaf.
- On this long path, some variable symbol R must repeat because of the pigeonhole principle.
- Replace the subtree under the second occurrence of R with the subtree under the first occurrence of R and still get a legal parse tree.

The Pumping Lemma for Context-Free Languages

Example (Surgery on parse trees)



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Proof.

Let G be a CFG for CFL A. Let b be the maximum number of symbols in the right-hand side of a rule.

- A node can have no more than b children. It means at most b^h leaves are within h steps of the start variable (the root of the parse tree).

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- If the height of the parse tree is at most h, the length of the string generated is at most b^h.
- Let V denote the number of variables in G, we set p, the pumping length, to be $b^{|V|+1}$.

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- The path has at least $|V|+2$ nodes and hence this path has at least $|V|+1$ variables.

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- Each occurrence of R has a subtree under it, generating a part of the string s.
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That establishes condition 1 of the lemma $\square$

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Proof.

For condition 2, we must be sure that v and y are not both ϵ.

- If they were, the parse tree obtained by substituting the smaller subtree for the larger would have fewer nodes than τ does and would still generate s.
- This result isn't possible because we had already chosen τ to be a parse tree for s with the smallest number of nodes.

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- The subtree where R generates vxy is at most $|V|+1$ high.
- A tree of this height can generate a string of length at most $b^{|V|+1} = p$.

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Example

Let $D = \{ww | w \in \{0,1\}^*\}$. Use the pumping lemma to show that D is not a CFL.

Outline

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- The languages that are recognizable by deterministic pushdown automata (DPDAs) are called deterministic context-free languages (DCFLs).
- Basic principle of determinism: at each step of its computation, the DPDA has at most one way to proceed according to its transition function.
- ϵ-moves is allowed in the DPDA' s transition function.
  - ϵ-input moves corresponding to $\delta(q, \epsilon, x)$;
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  - If a DPDA can make an ϵ-move in a certain situation, it is prohibited from making a move in that same situation that involves processing a symbol instead of ϵ.

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is not $\emptyset$.

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Example

$A = \{a^i b^j c^k | i \neq j$ or $j \neq k$ where $i, j, k \geq 0\}$ is a CFL but not a DCFL.