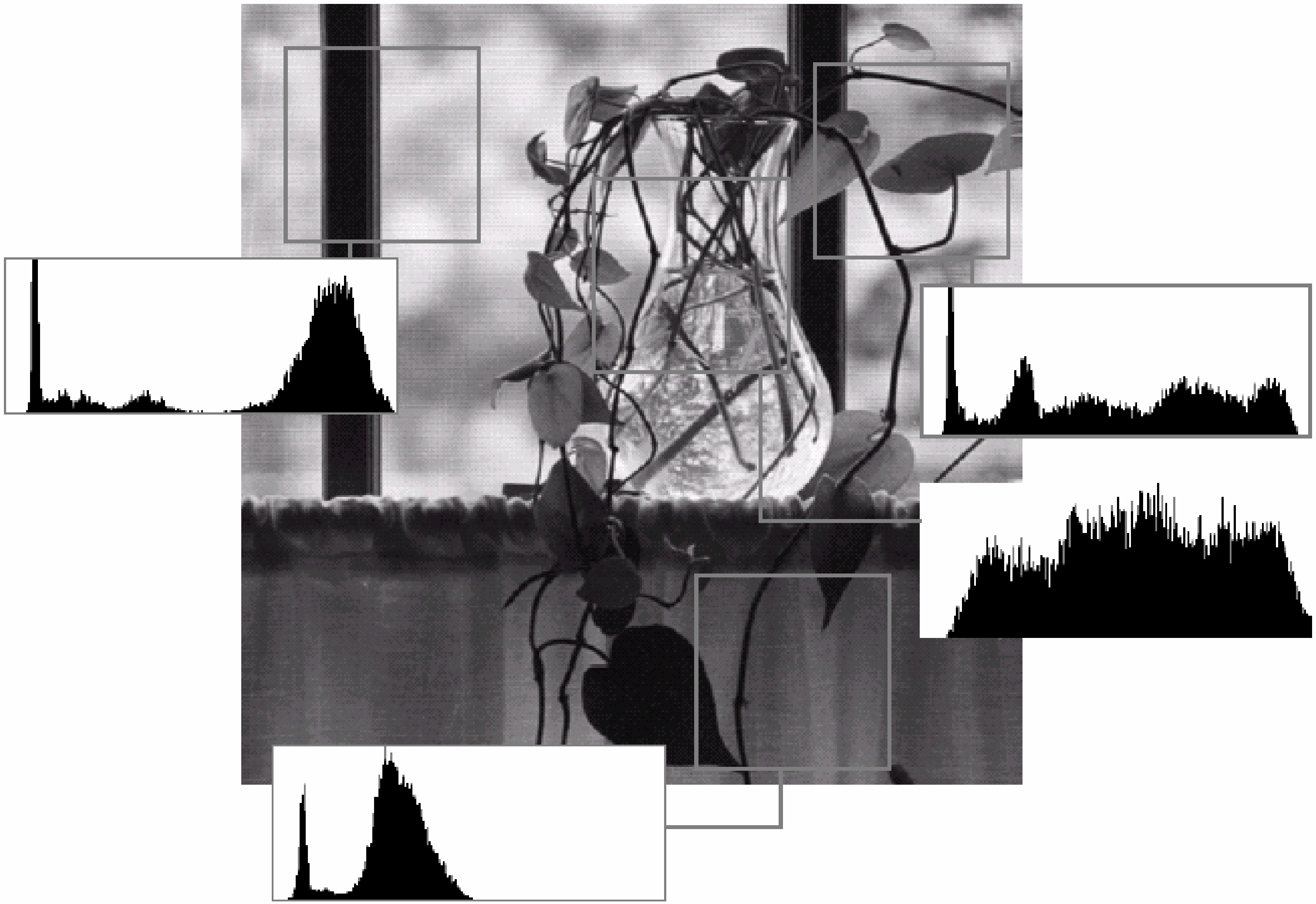
Chapter 7
Wavelets and Multiresolution Processing

- Background
- Multiresolution Expansions
- Wavelet Transforms in One Dimension
- Wavelet Transforms in Two Dimensions

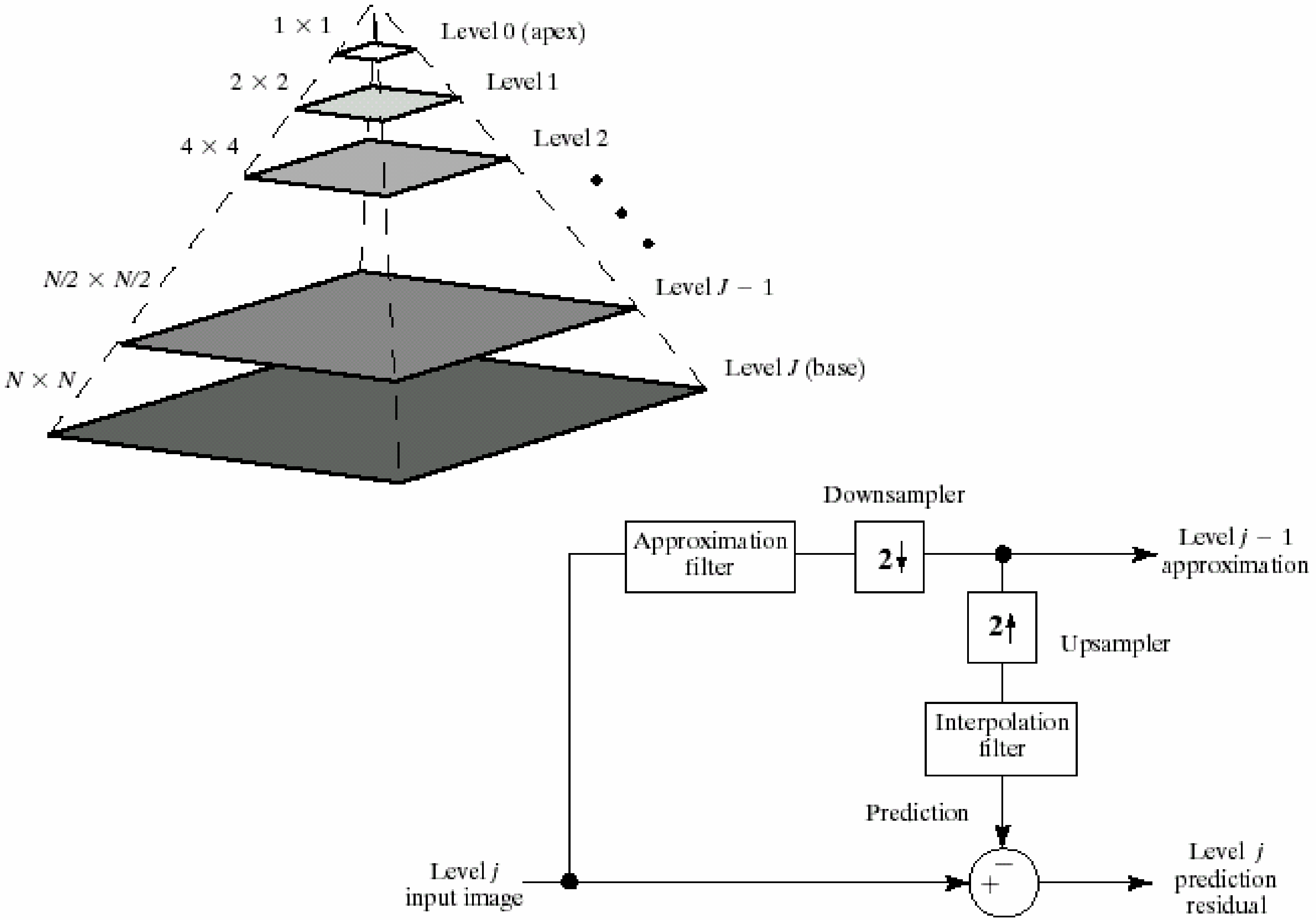
FIGURE 7.1 A natural image and its local histogram variations.



- Image Pyramids
- Subband Coding
- The Haar Transform

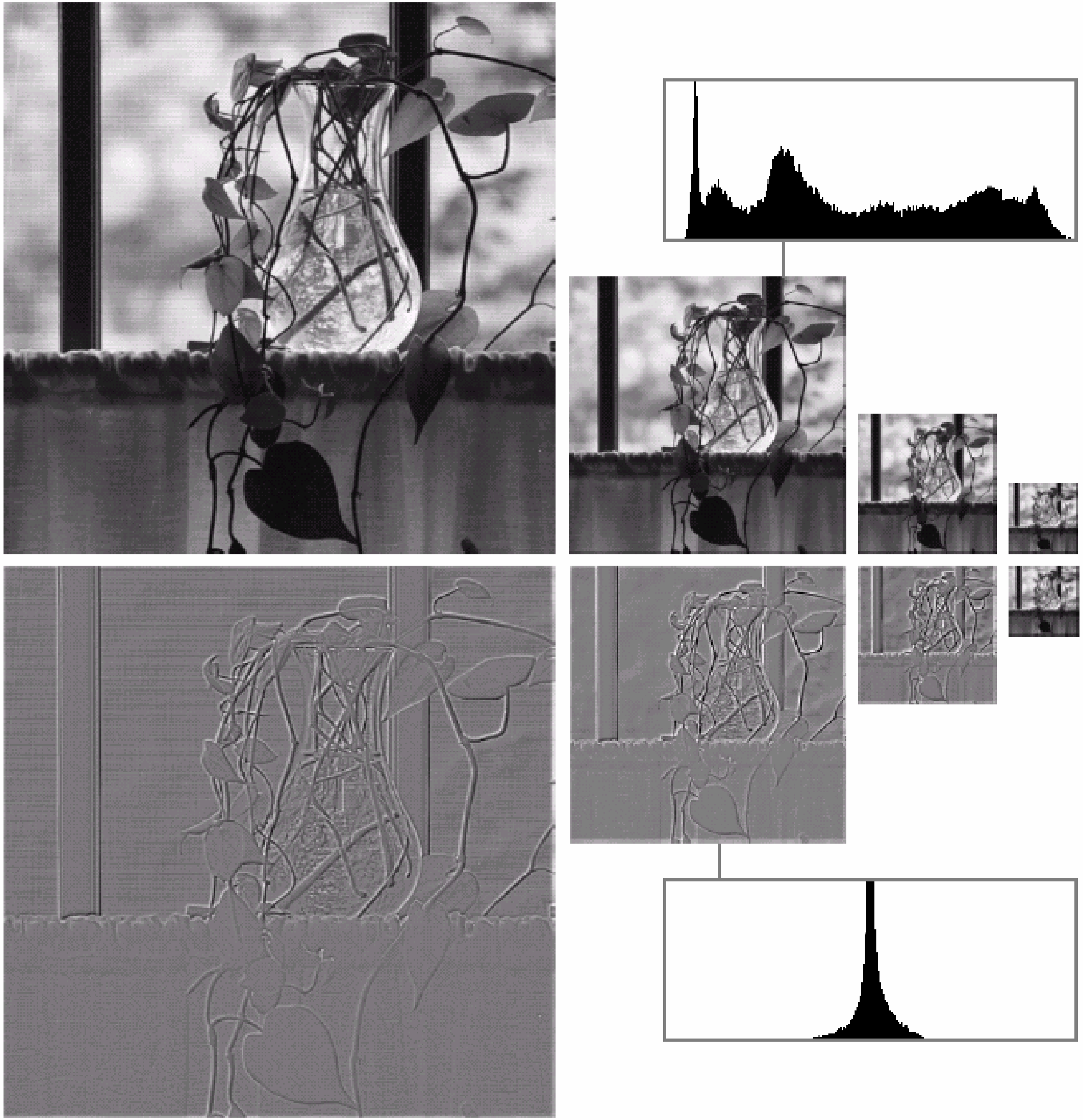
- The total number of elements in a P+1 level pyramid for P>0 is

$$N^2\left(1+\frac{1}{(4)^1}+\frac{1}{(4)^2}+\cdots+\frac{1}{(4)^P}\right)\leq\frac{4}{3}N^2$$



a b

FIGURE 7.2 (a) A pyramidal image structure and (b) system block diagram for creating it.



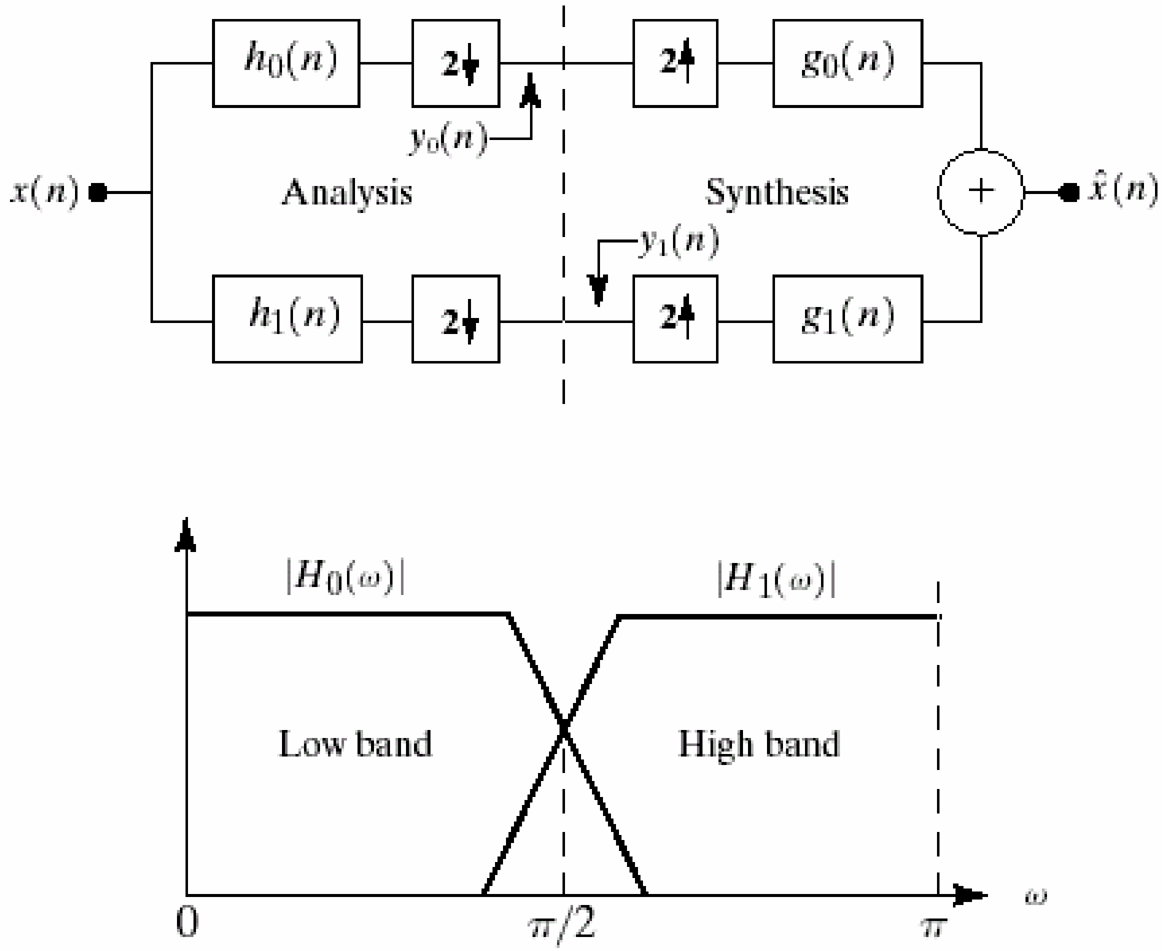
a
b

FIGURE 7.3 Two image pyramids and their statistics: (a) a Gaussian (approximation) pyramid and (b) a Laplacian (prediction residual) pyramid.

- Image Pyramids
- Subband Coding
- The Haar Transform

a
b

FIGURE 7.4 (a) A two-band filter bank for one-dimensional subband coding and decoding, and (b) its spectrum splitting properties.



- The Z-transform, a generalization of the discrete Fourier transform, is the ideal tool for studying discrete-time, sampled-data systems.
- The Z-transform of sequemce x(n) for n=0,1,2,…is

$$X(z)=\sum_{-\infty}^{\infty}x(n)z^{-n}$$

- Where z is a complex variable.

- Downsampling by a factor of 2 in the time domain corresponds to the simple Z-domain operation

$$x_{down}(n) = x(2n) \Leftrightarrow X_{down}(z) = \frac{1}{2}\left[X(z^{1/2}) + X\left(-z^{1/2}\right)\right] \quad (7.1\text{-}2)$$

- Upsampling-again by a factor of 2---is defined by the transform pair

$$x^{up}(n) = \begin{cases} x(n/2) & n = 0,2,4,\ldots \\ 0 & otherwise \end{cases} \Leftrightarrow X^{up}(z) = x(z^2) \quad (7.1\text{-}3)$$

- If sequence x(n) is downsampled and subsequently upsampled to yield $\hat{x}(n)$, Eqs.(7.1-2) and (7.1-3) combine to yield

$$\hat{X}(z) = \frac{1}{2}\left[X(z) + X(-z)\right]$$

where $\hat{x}(n) = Z^{-1}\left[\hat{X}(z)\right]$ is the downsampled-upsampled sequence.

- Its inverse Z-transform is

$$Z^{-1}\left[X(-z)\right] = (-1)^n x(n)$$

- We can express the system's output as

$$\hat{X}(z) = \frac{1}{2}G_0(z)\left[H_0(z)X(z) + H_0(-z)X(-z)\right]$$
$$+\frac{1}{2}G_1(z)\left[H_1(z)X(z) + H_1(-z)X(-z)\right]$$

- The output of filter $h_0(n)$ is defined by the transform pair

$$h_0(n) * x(n) = \sum_k h_0(n-k)x(k) \Leftrightarrow H_0(z)X(z)$$

- As with Fourier transform, convolution in the time (or spatial domain is equivalent to multiplication in the Z-domain.

$$\widehat{X}(z) = \frac{1}{2}\left[H_0(z)G_0(z) + H_1(z)G_1(z)\right]X(z)$$
$$+\frac{1}{2}\left[H_0(-z)G_0(z) + H_1(-z)G_1(z)\right]X(-z)$$

- For error-free reconstruction of the input, $\hat{x}(n) = x(n)$ and $\hat{X}(z) = X(z)$. Thus, we impose the following conditions:

$$H_0(-z)G_0(z) + H_1(-z)G_1(z) = 0$$

$$H_0(z)G_0(z) + H_1(z)G_1(z) = 2$$

To get

$$\begin{bmatrix} G_0(z) \\ G_1(z) \end{bmatrix} = \frac{2}{\det\left(H_m(z)\right)} \begin{bmatrix} H_1(-z) \\ -H_0(-z) \end{bmatrix}$$

Where $\det\left(H_m(z)\right)$ denotes the determinant of $H_m(z)$.

$$\det(H_m(z)) = \alpha z^{-(2k+1)}$$

- Letting $\alpha = 2$, and taking the inverse Z-transform, we get

$$g_0(n) = (-1)^n h_1(n)$$

$$g_1(n) = (-1)^{n+1} h_0(n)$$

- Letting $\alpha = -2$, and taking the inverse Z-transform, we get

$$g_0(n) = (-1)^{n+1} h_1(n)$$

$$g_1(n) = (-1)^n h_0(n)$$

Filter	QMF	CQF	Orthonormal
$H_0(z)$	$H_0^2(z) - H_0^2(-z) = 2$	$H_0(z)H_0(z^{-1}) +$ $H_0^2(-z)H_0(-z^{-1}) = 2$	$G_0(z^{-1})$
$H_1(z)$	$H_0(-z)$	$z^{-1}H_0(-z^{-1})$	$G_1(z^{-1})$
$G_0(z)$	$H_0(z)$	$H_0(z^{-1})$	$G_0(z)G_0(z^{-1}) +$ $G_0(-z)G_0(-z^{-1}) = 2$
$G_1(z)$	$-H_0(-z)$	$zH_0(-z)$	$-z^{-2K+1}G_0(-z^{-1})$

TABLE 7.1 Perfect reconstruction filter families.

- Three general solution:
  - Quadrature mirror filters (OMFs)
  - Conjugate quadrature filters (CQFs)
  - Orthonormal

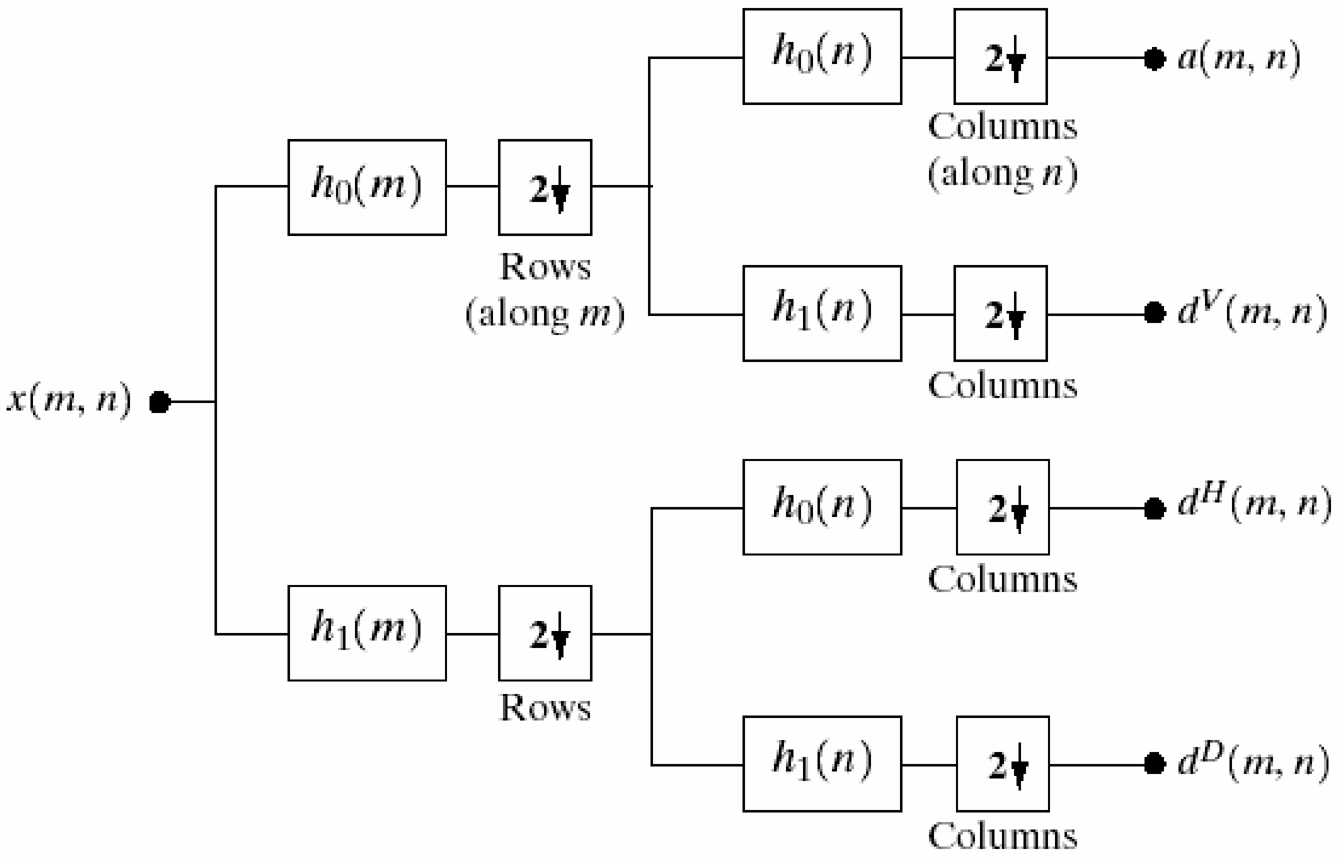
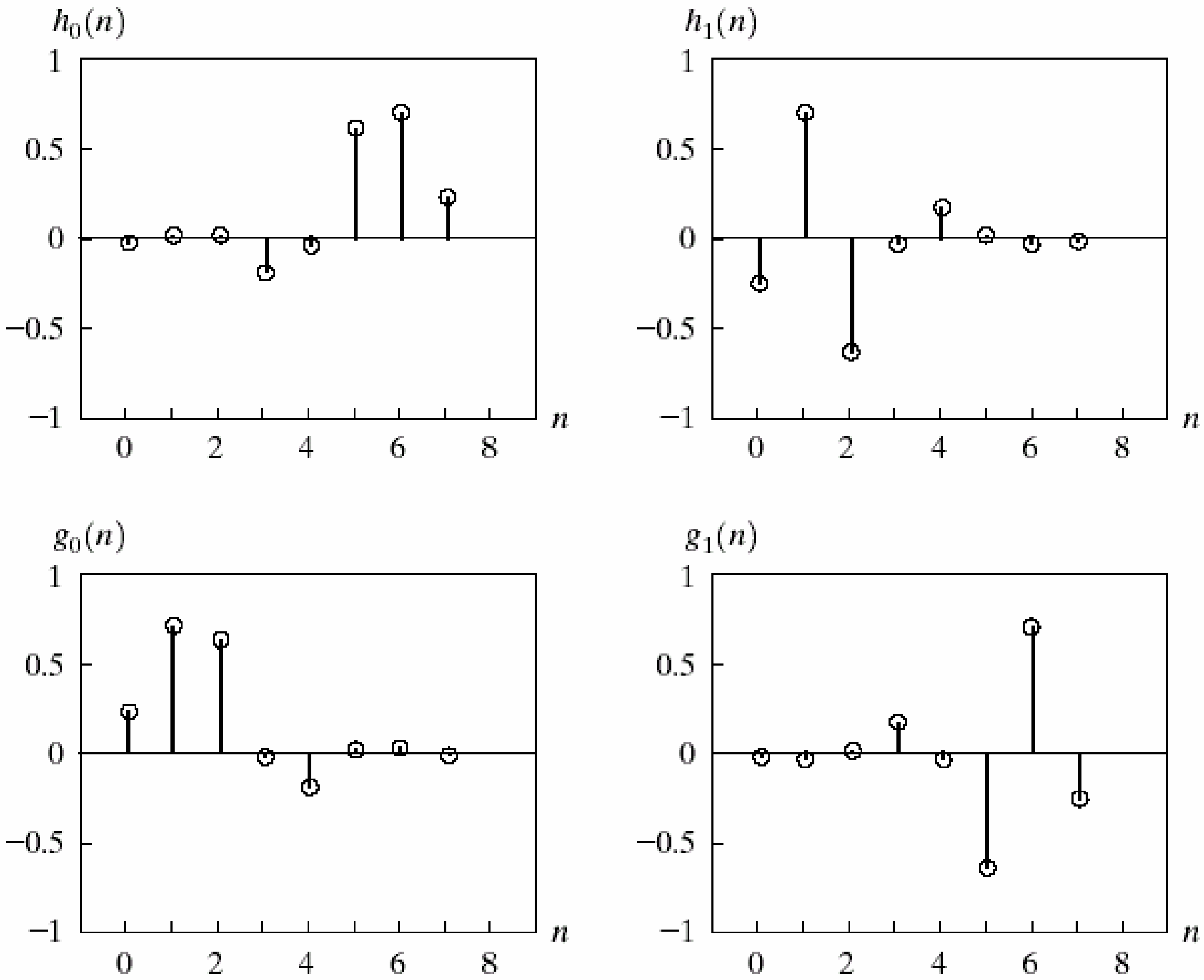


FIGURE 7.5 A two-dimensional, four-band filter bank for subband image coding.

FIGURE 7.6 The impulse responses of four 8-tap Daubechies orthonormal filters.



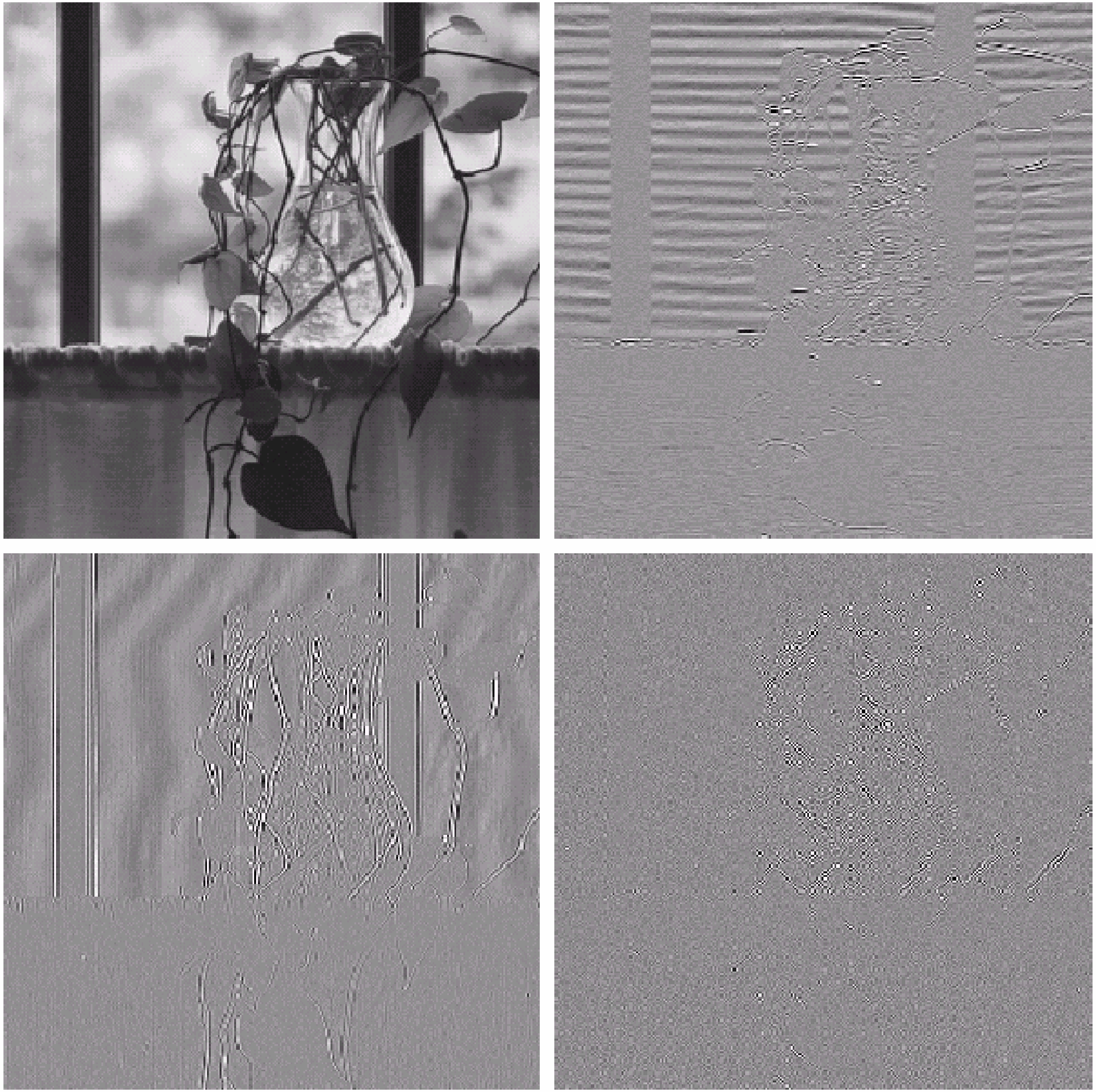


FIGURE 7.7 A four-band split of the vase in Fig. 7.1 using the subband coding system of Fig. 7.5.

- Image Pyramids
- Subband Coding
- The Haar Transform

- The Haar transform can be expressed in matrix form

$$T = HFH^T$$

- Where
  - F is an N*N image matrix,
  - H is an N*N transformation matrix,
  - T is the resulting N*N transform.

- For the Haar transform, transformation matrix H contains the Haar basis functions, $h_k(z)$ .They are defined over the continuous, closed interval $z \in [0,1]$ for k=0,1,2,…,N-1, where $N = 2^n$.
- To generate H, we define the integer k such that

$$k = 2^P + q - 1$$

where $0 \le p \le n-1$, $q = 0$ or 1 for $p = 0$.

$$1 \le q \le 2^p \quad for \quad p \ne 0$$

- Then the Haar basis functions are

$$h_0(z) = h_{00}(z) = \frac{1}{\sqrt{N}}, \qquad z \in [0,1]$$

- and

$$h_k(z) = h_{pq}(z) = \frac{1}{\sqrt{N}} \begin{cases} 2^{p/2} & (q-1)/2^p \le z < (q-0.5)/2^p \\ -2^{p/2} & (q-0.5)/2^p \le z < q/2^p \\ 0 & otherwise, z \in [0,1] \end{cases}$$

- The ith row of an N*N Haar transformation matrix contains the elements of

$$h_i(z) \quad for \quad z = 0/N, 1/N, 2/N, ..., (N-1)/N.$$

- If N=4, for example k,q, and p assume the values

k	p	q
0	0	0
1	0	1
2	1	1
3	1	2

- The 4*4 transformation matrix, H_4, is

$$H_4 = \frac{1}{\sqrt{4}} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ \sqrt{2} & -\sqrt{2} & 0 & 0 \\ 0 & 0 & \sqrt{2} & -\sqrt{2} \end{bmatrix}$$

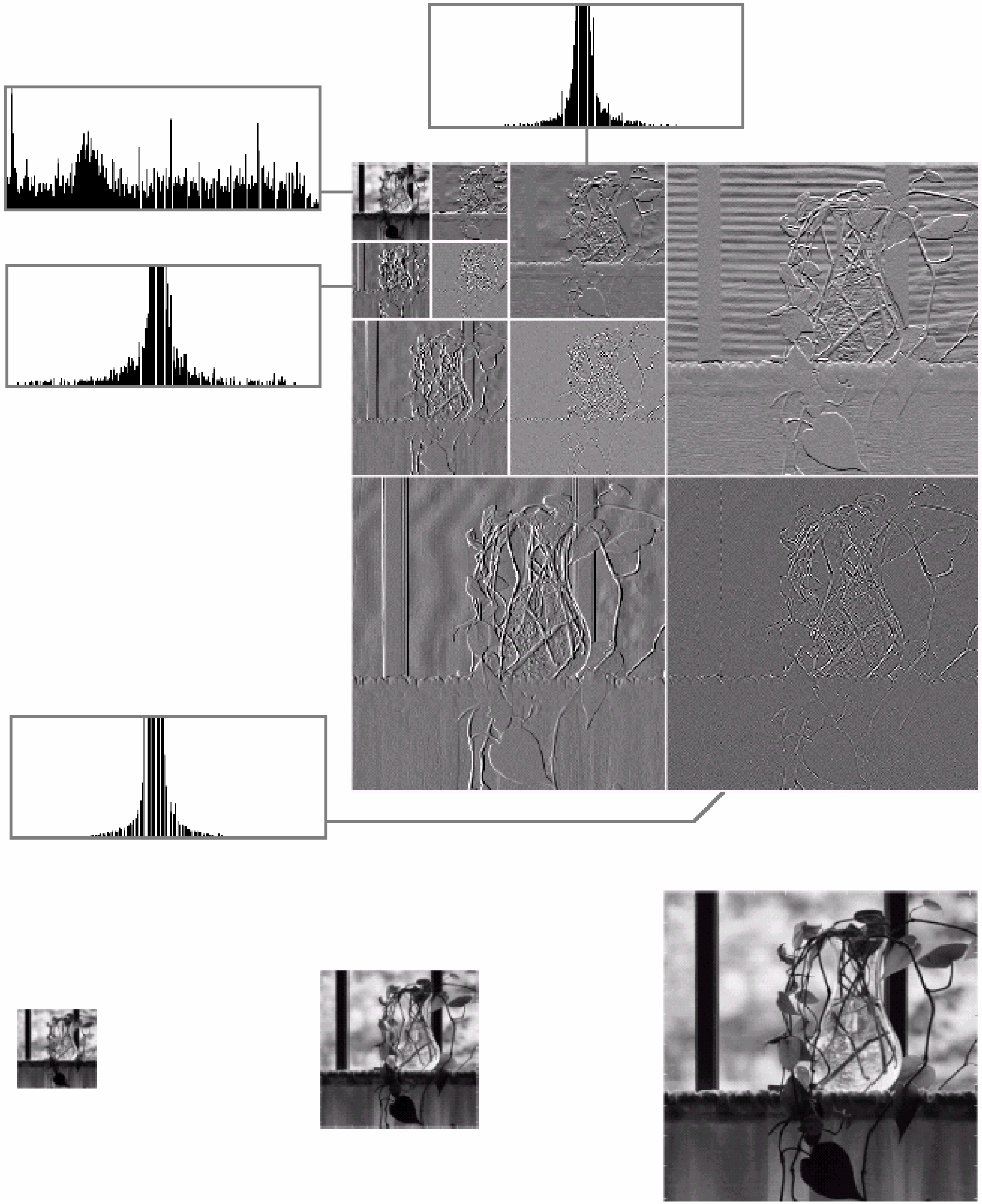


FIGURE 7.8 (a) A discrete wavelet transform using Haar basis functions. Its local histogram variations are also shown; (b)–(d) Several different approximations (64×64, 128×128, and 256×256) that can be obtained from (a).

- Background
- Multiresolution Expansions
- Wavelet Transforms in One Dimension
- Wavelet Transforms in Two Dimensions

- Series Expansion
- Scaling Functions
- Wavelet Functions

- A signal of function f(x) can often be better analyzed as a linear combination of expansion functions

$$f(x) = \sum_k \alpha_k \varphi_k(x)$$

- k is an interger index of the finite or infinite sum;
- α_k are real-valued expansion coefficients;
- $\varphi_k(x)$ are real-valued expansion functions.

- These coefficients are computed by taking the integral inner products of the dual $\tilde{\varphi}_k(x)$'s and function f(x). That is

$$\alpha_k = \left\langle \tilde{\varphi}_k(x), f(x) \right\rangle = \int \tilde{\varphi}_k^*(x) f(x) dx$$

- Series Expansion
- Scaling Functions
- Wavelet Functions

- The set of expansion functions composed of integer translations and binary scaling of the real, square-integrable function $\varphi(x)$; that is, the set $\{\varphi_{j,k}(x)\}$ where

$$\varphi_{j,k}(x) = 2^{j/2}\varphi\left(2^j x - k\right)$$

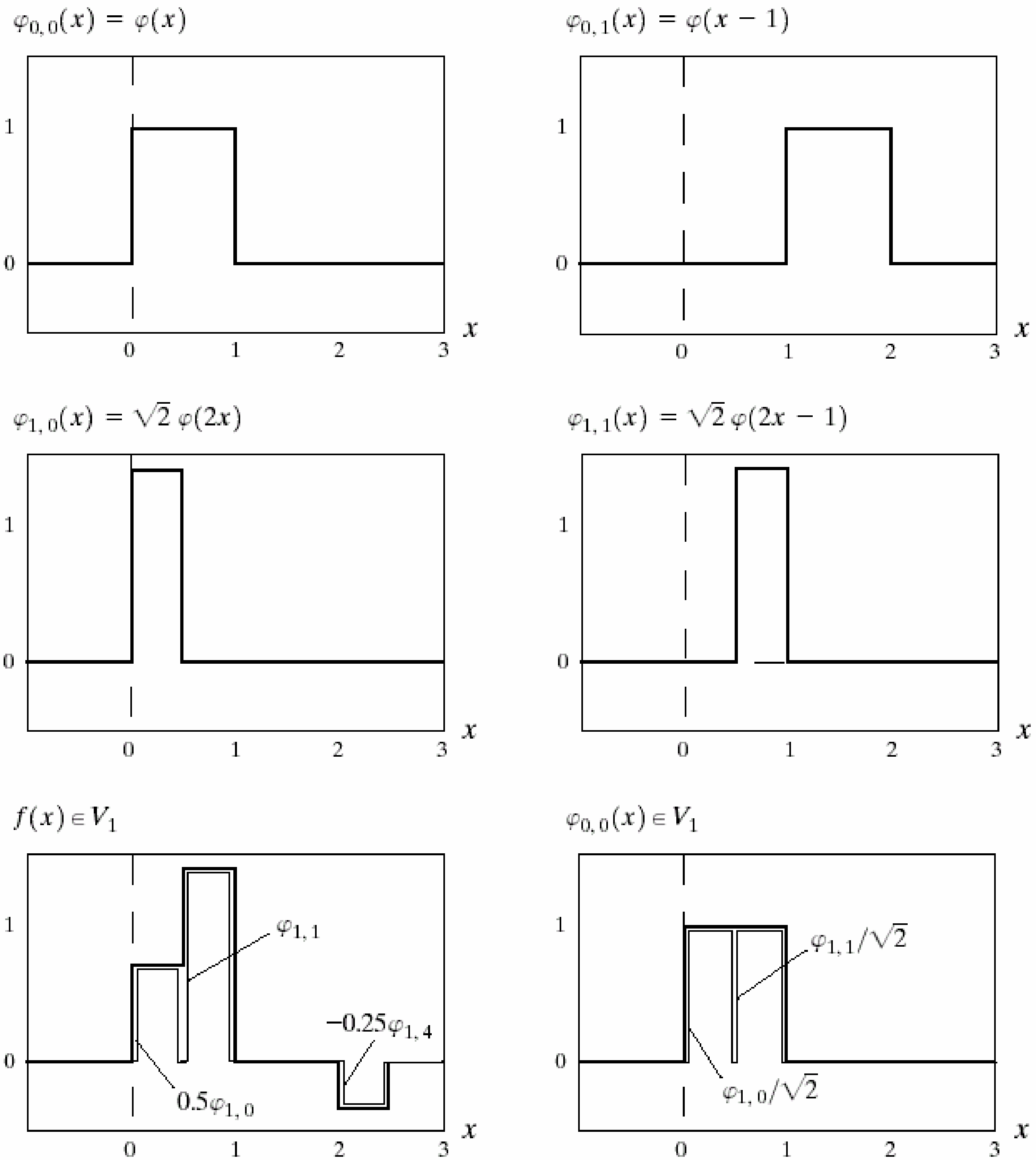
$$V_{j_0} = \overline{\underset{k}{Span}\left\{\varphi_{j_0,k}(x)\right\}}$$

- If $f(x) \in V_{j_0}$, it can be written

$$f(x) = \sum_k \alpha_k \varphi_{j_0,k}(x)$$

- We will denote the subspace spanned over k for any j as

$$V_j = \overline{\underset{k}{Span}\left\{\varphi_{j,k}(x)\right\}}$$

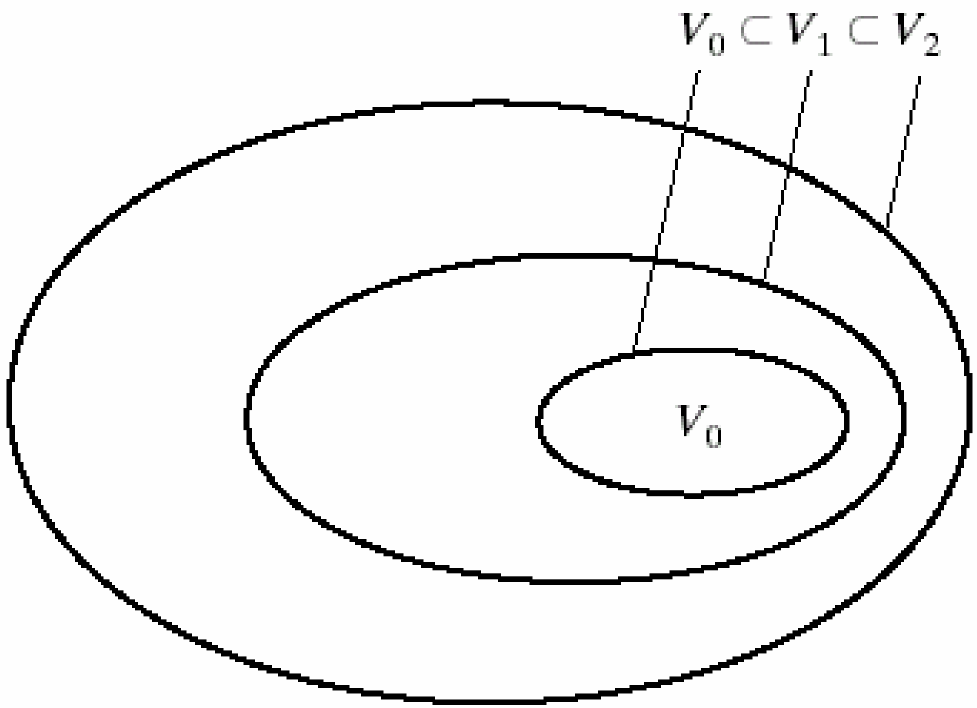


a b
c d
e f

FIGURE 7.9 Haar scaling functions in V_0 in V_1.

- The simple scaling function in the preceding example obeys the four fundamental requirements of multiresolution analysis:
  - MRA Requirement 1: The scaling function is orthogonal to its integer translates;
  - MRA Requirement 2: The subspaces spanned by the scaling function at low scales are nested within those spanned at higher scales.

FIGURE 7.10 The nested function spaces spanned by a scaling function.



- MRA Requirement 3: The only function that is common to all V_j is $f(x)=0$.
- MRA Requirement 4: Any function can be represented with arbitrary precision.

- Series Expansion
- Scaling Functions
- Wavelet Functions

- Given a scaling function that meets the MRA requirements of the previous section, we can define a wavelet function $\psi(x)$ that, together with its integer translates and binary scaling, spans the difference between any two adjacent scaling subspaces, V_j and V_{j+1}. We define the set $\{\psi_{j,k}(x)\}$ of wavelets

$$\left\{\psi_{j,k}(x) = 2^{j/2}\psi(2^j x - k)\right\}$$

- As with scaling functions, we write

$$W_j = \overline{\underset{k}{Span}\{\psi_{j,k}(x)\}}$$

- And note that if $f(x) \in W_j$

$$f(x) = \sum_k \alpha_k \psi_{j,k}(x)$$

- The scaling and wavelet function subspaces are related by

$$V_{j+1} = V_j \oplus W_j$$

- We can now express the space of all measurable, square-integrable functions as

$$L^2(R) = V_0 \oplus W_0 \oplus W_1 \oplus \dots$$

- or

$$L^2(R) = V_1 \oplus W_1 \oplus W_2 \oplus \dots$$

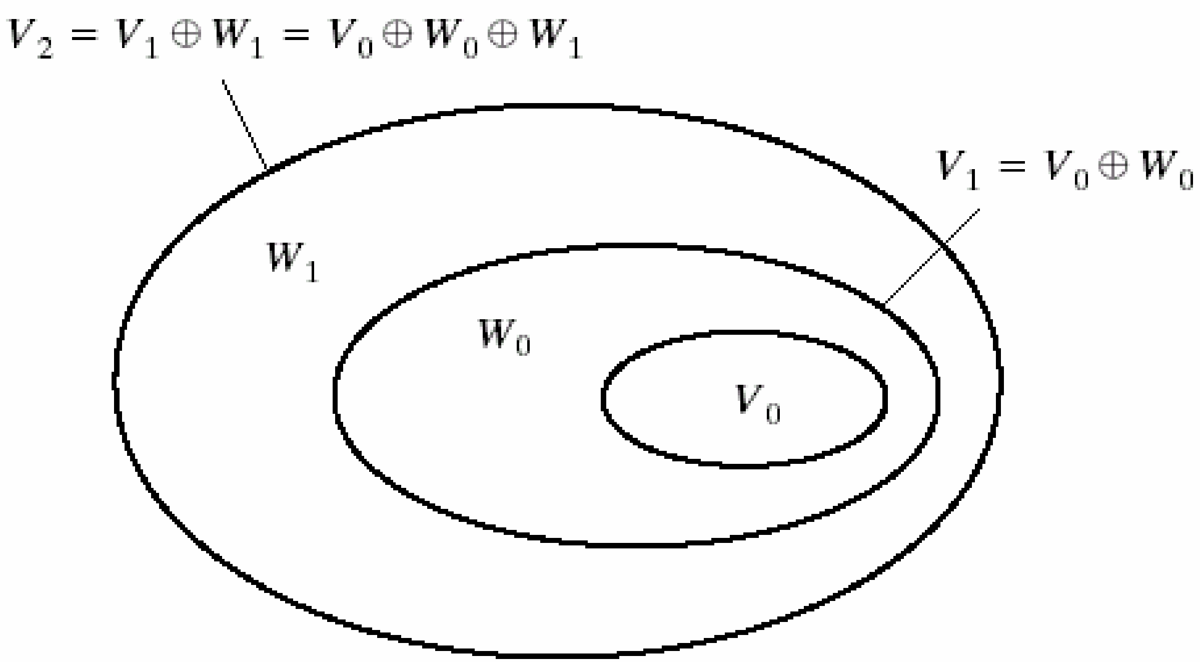
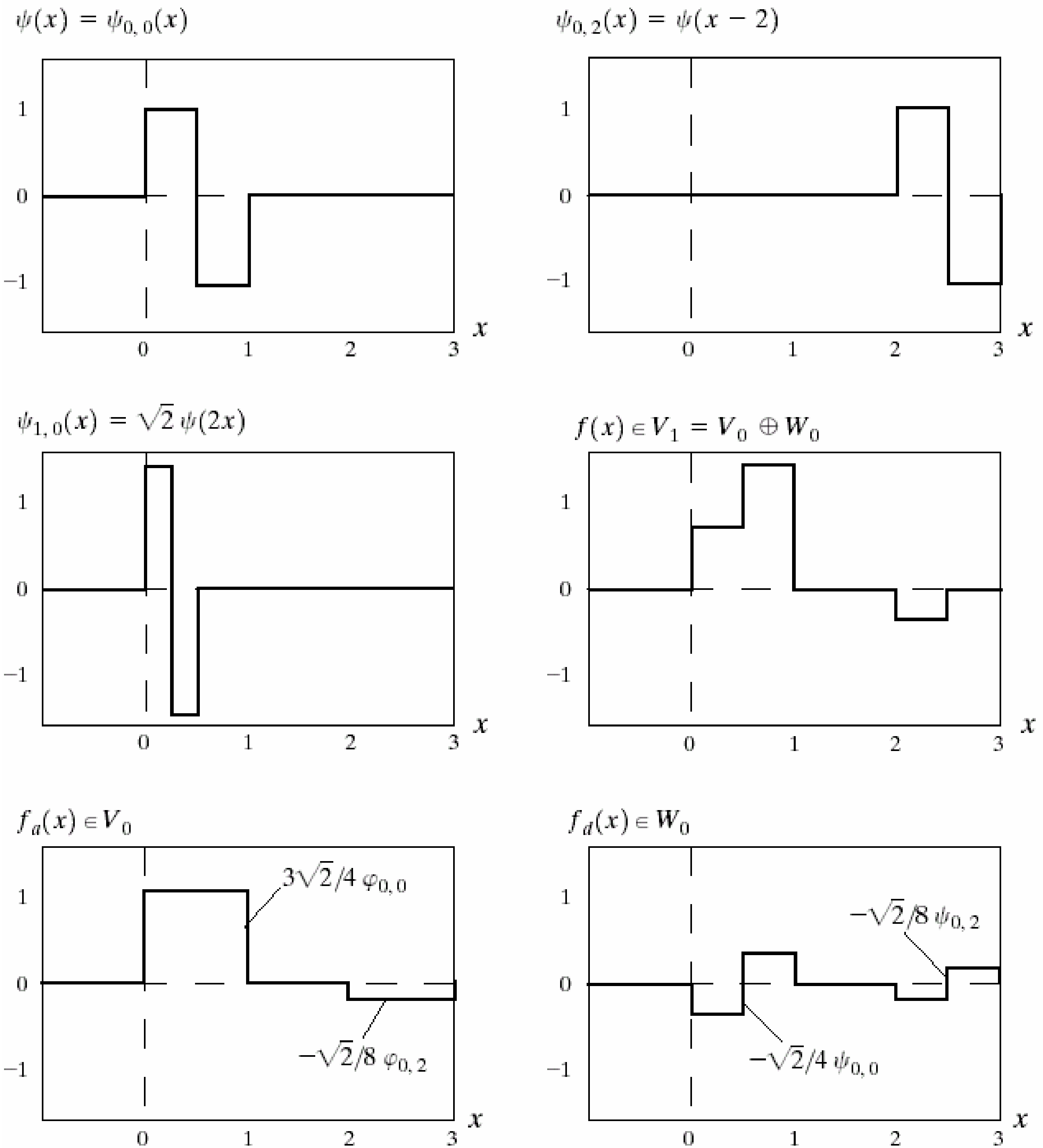


FIGURE 7.11 The relationship between scaling and wavelet function spaces.

- The Haar wavelet function is

$$\psi(x) = \begin{cases} 1 & 0 \le x < 0.5 \\ -1 & 0.5 \le x < 1 \\ 0 & elsewhere \end{cases}$$



a b
c d
e f

FIGURE 7.12 Haar wavelet functions in W_0 and W_1.

- Background
- Multiresolution Expansions
- Wavelet Transforms in One Dimension
- Wavelet Transforms in Two Dimensions

- The Wavelet Series Expansions
- The Discrete Wavelet Transform
- The Continuous Wavelet Transform

- Defining the wavelet series expansion of function $f(x) \in L^2(R)$ relative to wavelet $\psi(x)$ and scaling function $\varphi(x)$. f(x) can be written as

$$f(x) = \sum_k c_{j_0}(k)\varphi_{j_0,k}(x) + \sum_{j=j_0}^{\infty} d_j(k)\psi_{j,k}(x)$$

- $c_{j_0}(k)'s$: the approximation or scaling coefficients;
- $d_j(k)'s$: the detail or wavelet coefficients.

- If the expansion functions form an orthonormal basis or tight frame, the expansion coefficients are calculated as

$$c_{j_0}(k) = \left\langle f(x), \varphi_{j_0,k}(x) \right\rangle = \int f(x)\varphi_{j_0,k}(x)dx$$

- and

$$d_j(k) = \left\langle f(x), \psi_{j,k}(x) \right\rangle = \int f(x)\psi_{j,k}(x)dx$$

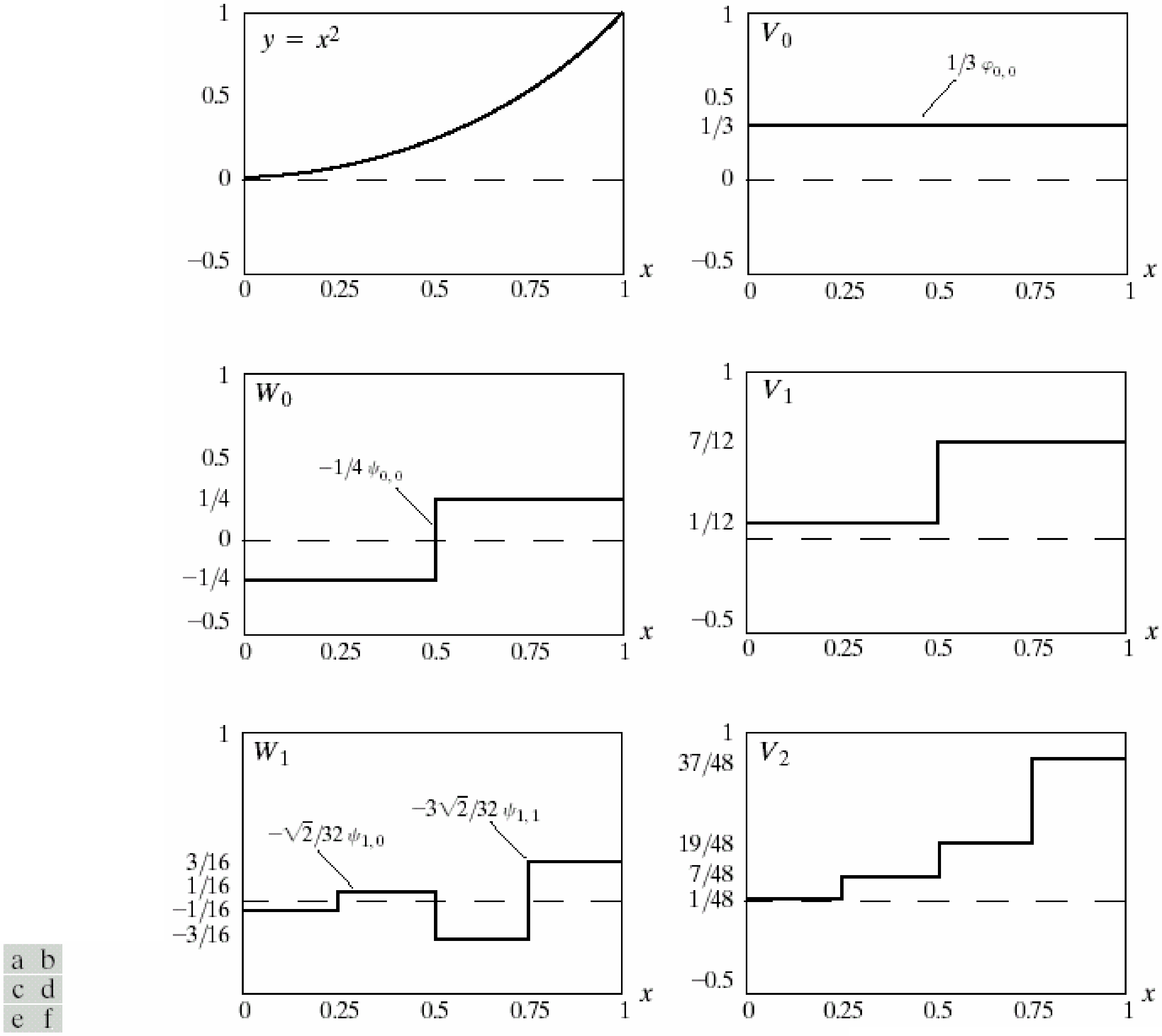


FIGURE 7.13 A wavelet series expansion of $y = x^2$ using Haar wavelets.

- The Wavelet Series Expansions
- The Discrete Wavelet Transform
- The Continuous Wavelet Transform

- If the function being expanded is a sequence of numbers, like samples of a continuous function f(x), the resulting coefficients are called the discrete wavelet transform(DWT) of f(x).

$$W_{\varphi}(j_0,k)=\frac{1}{\sqrt{M}}\sum_x f(x)\varphi_{j_0,k}(x)$$

$$W_{\psi}(j,k)=\frac{1}{\sqrt{M}}\sum_x f(x)\psi_{j,k}(x)$$

- and

$$f(x)=\frac{1}{\sqrt{M}}\sum_k W_{\varphi}(j_0,k)\varphi_{j_0,k}(x)+\frac{1}{\sqrt{M}}\sum_{j=j_0}^{\infty}\sum_k W_{\psi}(j,k)\psi_{j,k}(x)$$

- Consider the discrete function of four points:

  f(0)=1, f(1)=4, f(2)=-3, and f(3)=0

- Since M=4, J=2 and, with j_0=0, the summations are performed over

  x=0,1,2,3,

  j=0,1, and

  k=0 for j=0

  or k=0,1 for j=1.

- We find that

$$W_\varphi(0,0)=\frac{1}{2}\sum_{x=0}^{3}f(x)\varphi_{0,0}(x)=\frac{1}{2}\left[1\cdot 1+4\cdot 1-3\cdot 1+0\cdot 1\right]=1$$

$$W_\psi(0,0)=\frac{1}{2}\left[1\cdot 1+4\cdot 1-3\cdot(-1)+0\cdot(-1)\right]=4$$

$$W_\psi(1,0)=\frac{1}{2}\left[1\cdot\sqrt{2}+4\cdot(-\sqrt{2})-3\cdot 0+0\cdot 0\right]=-1.5\sqrt{2}$$

$$W_\psi(1,1)=\frac{1}{2}\left[1\cdot 0+4\cdot 0-3\cdot\sqrt{2}+0\cdot(-\sqrt{2})\right]=-1.5\sqrt{2}$$

$$f(x)=\frac{1}{2}\left[W_{\varphi}(0,0)\varphi_{0,0}(x)+W_{\psi}(0,0)\psi_{0,0}(x)+W_{\psi}(1,0)\psi_{1,0}(x)+W_{\psi}(1,1)\psi_{1,1}(x)\right]$$

- For x=0,1,2,3. If x=0, for instance,

$$f(0)=\frac{1}{2}\left[1\cdot 1+4\cdot 1-1.5\sqrt{2}\cdot(\sqrt{2})-1.5\sqrt{2}\cdot 0\right]=1$$

- The Wavelet Series Expansions
- The Discrete Wavelet Transform
- The Continuous Wavelet Transform

- The continuous wavelet transform of a continuous, square-integrable function, f(x), relative to a real-valued wavelet, $\psi(x)$,is

$$W_{\psi}(s,\tau)=\int_{-\infty}^{\infty} f(x)\psi_{s,\tau}(x)dx$$

- Where

$$\psi_{s,\tau}(x)=\frac{1}{\sqrt{s}}\psi\left(\frac{x-\tau}{s}\right)$$

- And s and τ are called scale and translation parameters.

- Given $W_\psi(s,\tau)$, f(x) can be obtained using the inverse continuous wavelet transform

$$f(x) = \frac{1}{C_\psi} \int_0^\infty \int_{-\infty}^\infty W_\psi(s,\tau) \frac{\psi_{s,\tau}(x)}{s^2} d\tau ds$$

- Where

$$C_\psi = \int_{-\infty}^\infty \frac{|\Psi(u)|^2}{|u|} du$$

- And $\Psi(u)$ is the Fourier transform of $\psi(x)$.

- The Mexican hat wavelet

$$\psi(x) = \left(\frac{2}{\sqrt{3}} \pi^{-1/4} \right) (1 - x^2) e^{-x^2/2}$$

a b
c d

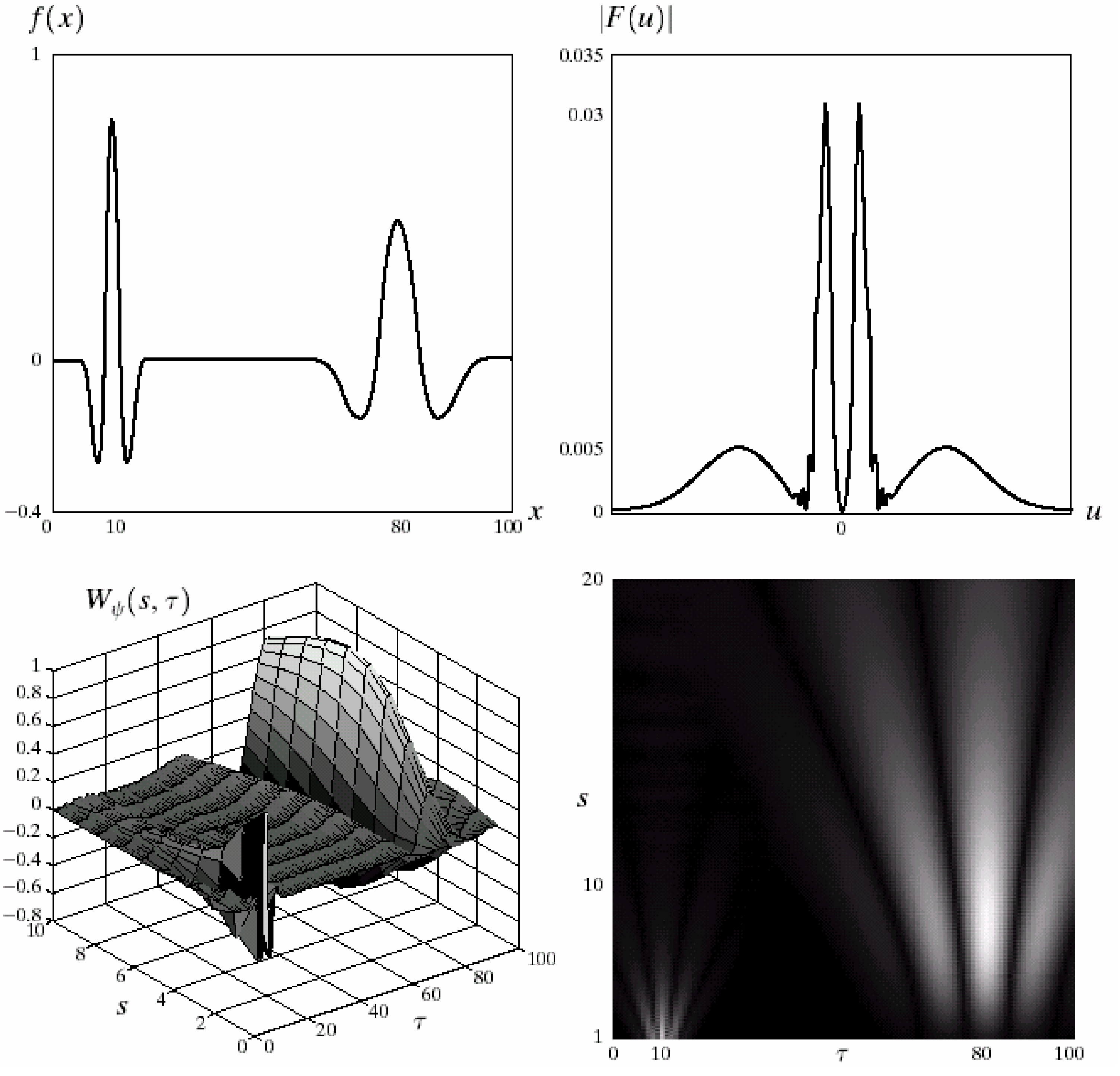


FIGURE 7.14 The continuous wavelet transform (c and d) and Fourier spectrum (b) of a continuous one-dimensional function (a).

- Background
- Multiresolution Expansions
- Wavelet Transforms in One Dimension
- Wavelet Transforms in Two Dimensions

- In two dimensions, a two-dimensional scaling function, $\varphi(x, y)$, and three two-dimensional wavelet, $\psi^{H}(x, y)$, $\psi^{V}(x, y)$ and $\psi^{V}(x, y)$, are required.

- Excluding products that produce one-dimensional results, like $\varphi(x)\psi(x)$, the four remaining products produce the separable scaling function

$$\varphi(x,y) = \varphi(x)\varphi(y)$$

- And separable, “directionally sensitive” wavelets

$$\psi^{H}(x,y) = \psi(x)\varphi(y)$$

$$\psi^{V}(x,y) = \varphi(x)\psi(y)$$

$$\psi^{D}(x,y) = \psi(x)\psi(y)$$

- The scaled and translated basis functions:

$$\varphi_{j,m,n}(x,y) = 2^{j/2}\varphi(2^j x - m, 2^j y - n)$$

$$\psi^i{}_{j,m,n}(x,y) = 2^{j/2}\psi(2^j x - m, 2^j y - n), \qquad i = \{H, V, D\}$$

- The discrete wavelet transform of function f(x,y) of size M*N is then

$$W_{\varphi}(j_0, m, n) = \frac{1}{\sqrt{MN}} \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x, y) \varphi_{j_0, m, n}(x, y)$$

$$W_{\psi}^{i}(j, m, n) = \frac{1}{\sqrt{MN}} \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x, y) \psi^{i}{}_{j, m, n}(x, y) \qquad i = \{H, V, D\}$$

- Given the W_φ and W_ψ^i , f(x,y) is obtained via the inverse discrete wavelet transform

$$f(x,y) = \frac{1}{\sqrt{MN}} \sum_m \sum_n W_\varphi(j_0, m, n) \varphi_{j_0,m,n}(x,y)$$
$$+ \frac{1}{\sqrt{MN}} \sum_{i=H,V,D} \sum_{j=j_0}^{\infty} \sum_m \sum_n W_\psi^i(j,m,n) \psi^i_{j,m,n}(x,y)$$

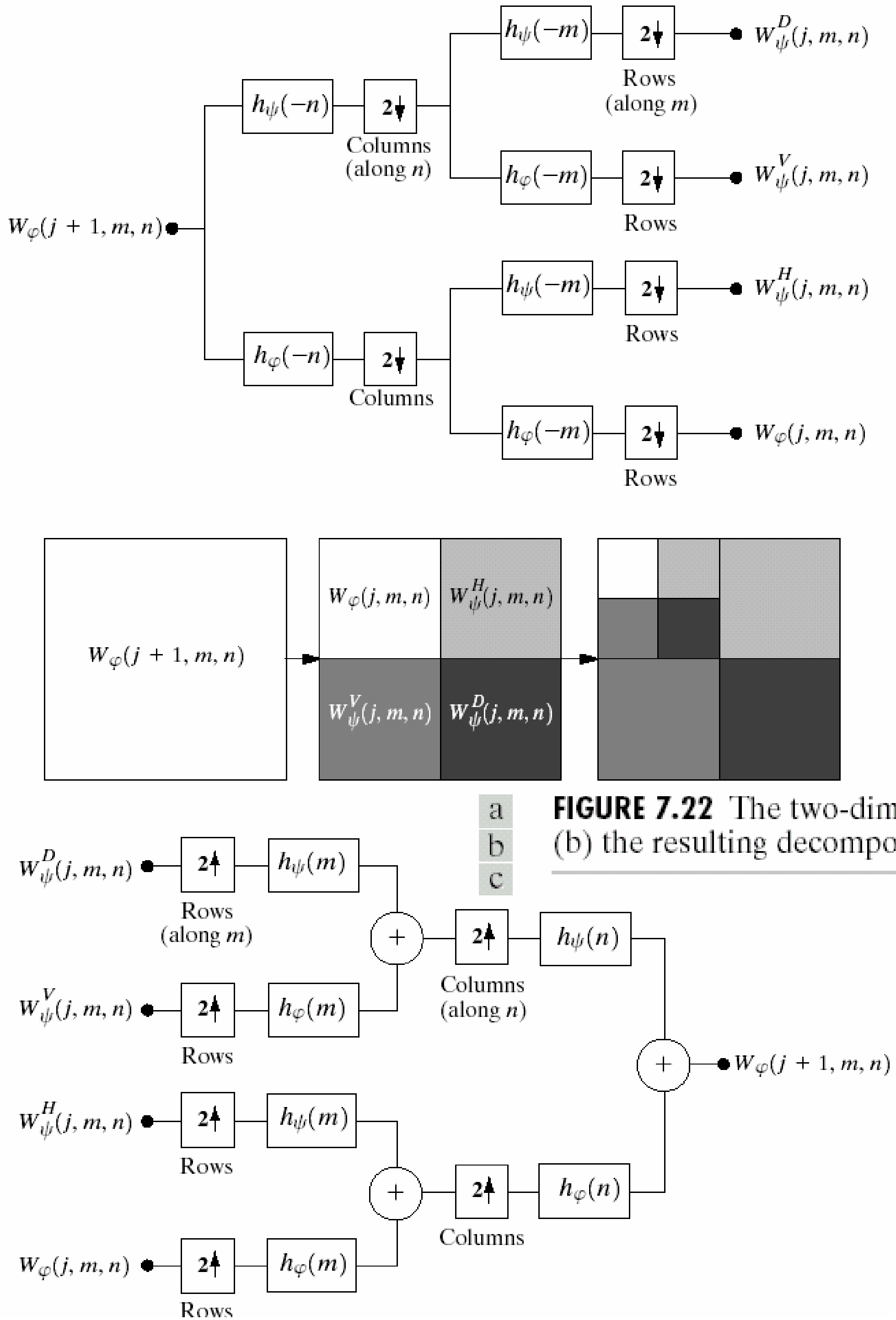
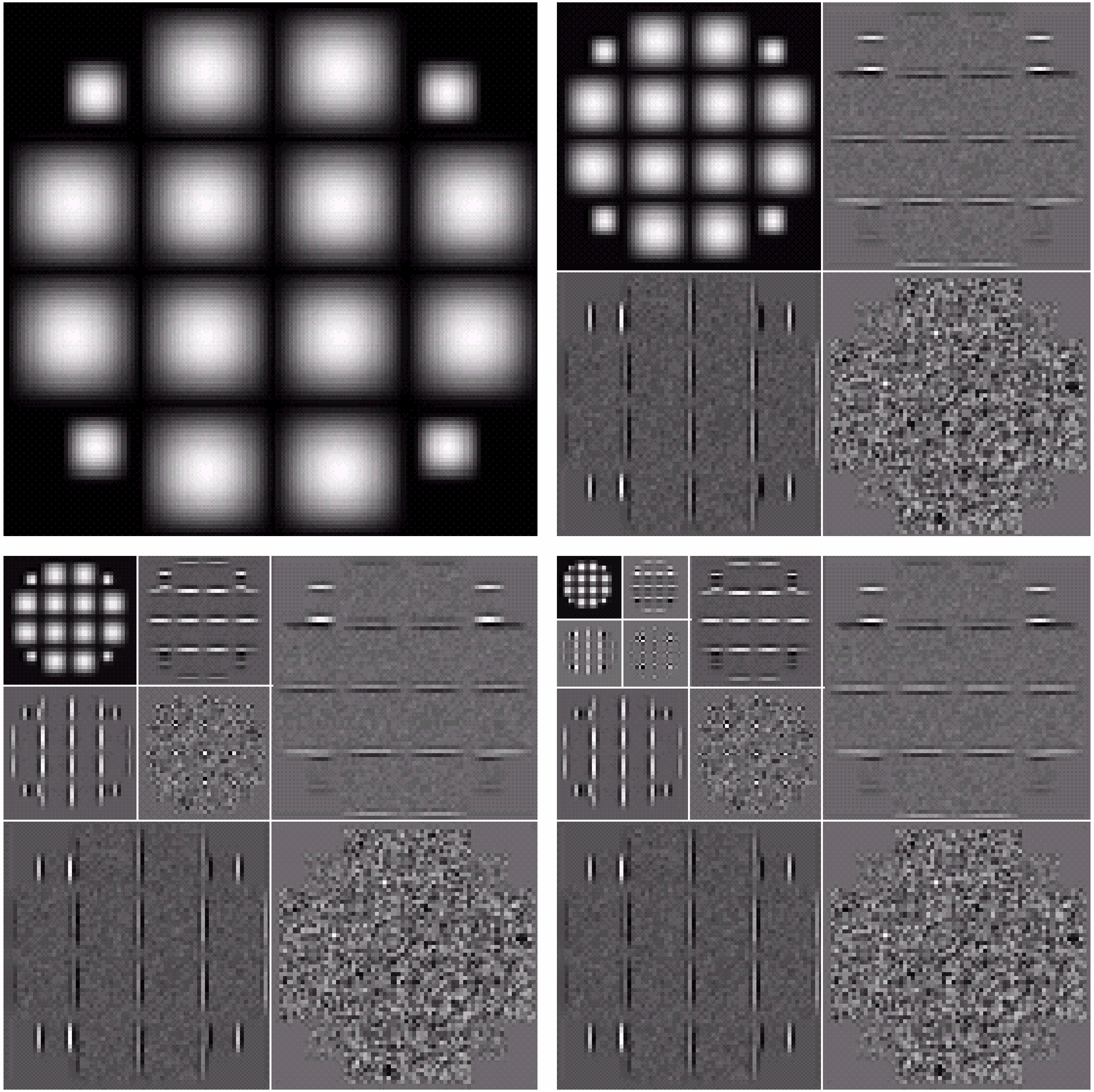
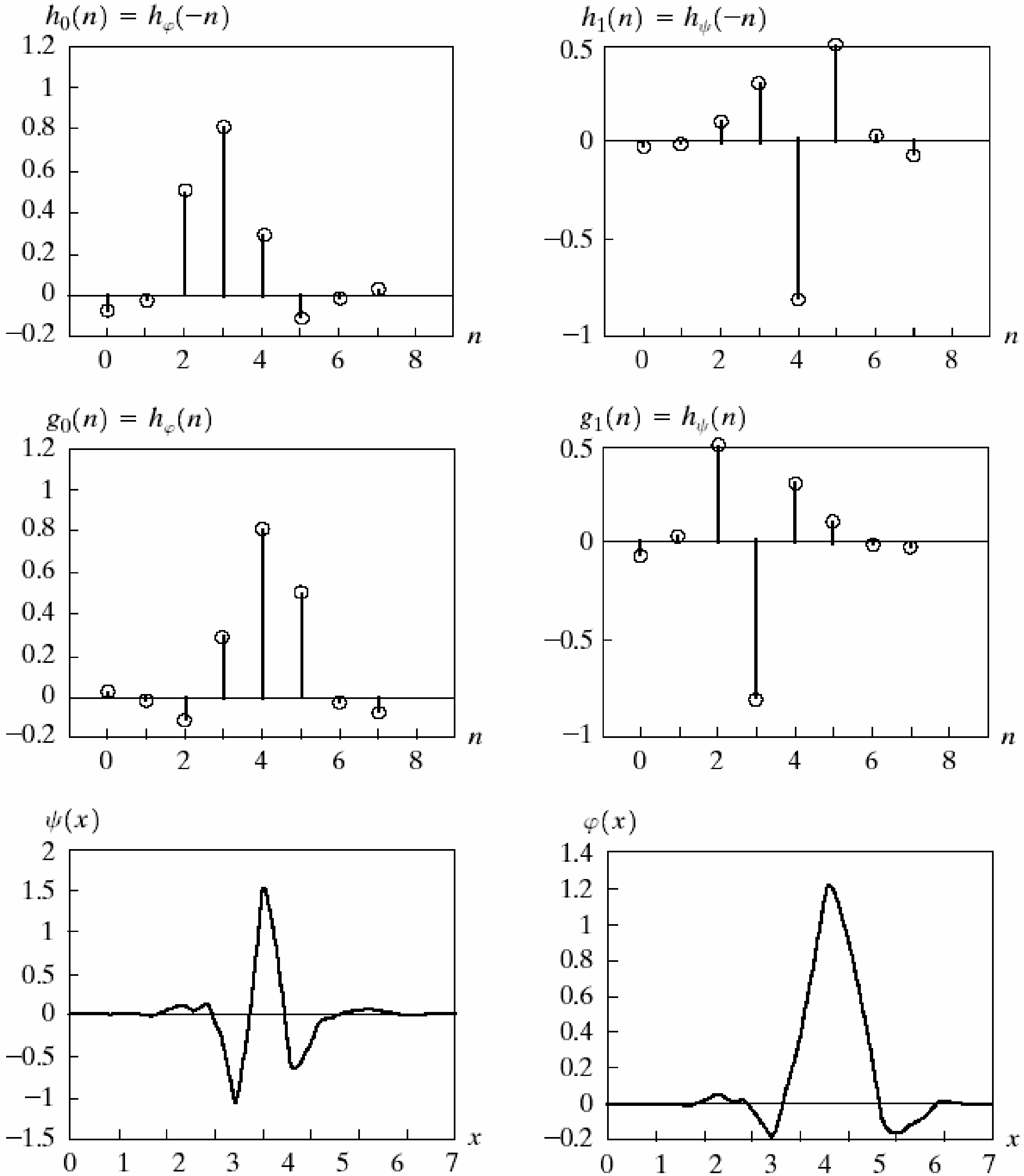


FIGURE 7.22 The two-dimensional fast wavelet transform: (a) the analysis filter bank; (b) the resulting decomposition; and (c) the synthesis filter bank.



a	b
c	d

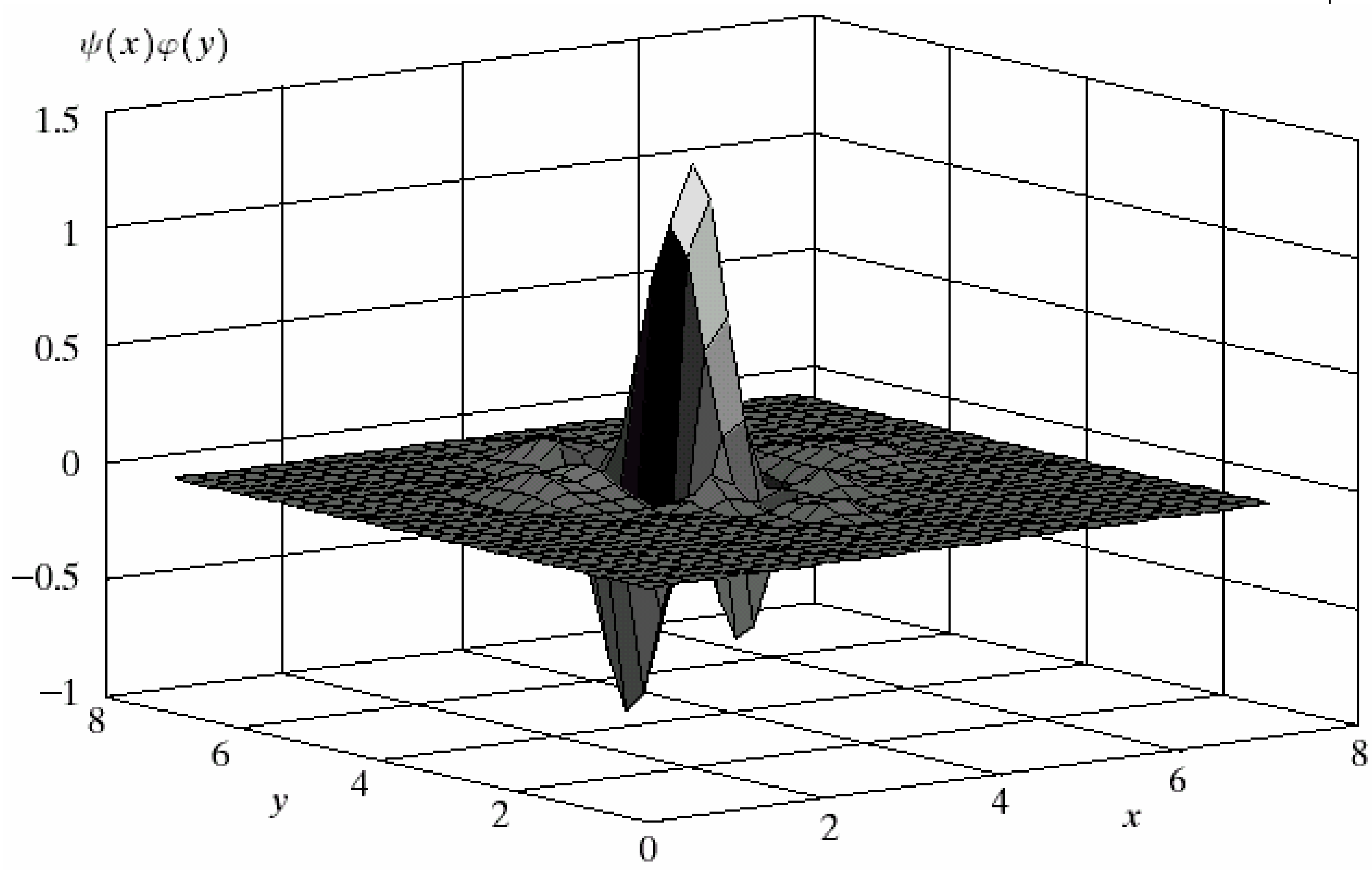
FIGURE 7.23 A three-scale FWT.



a b
c d
e f
g

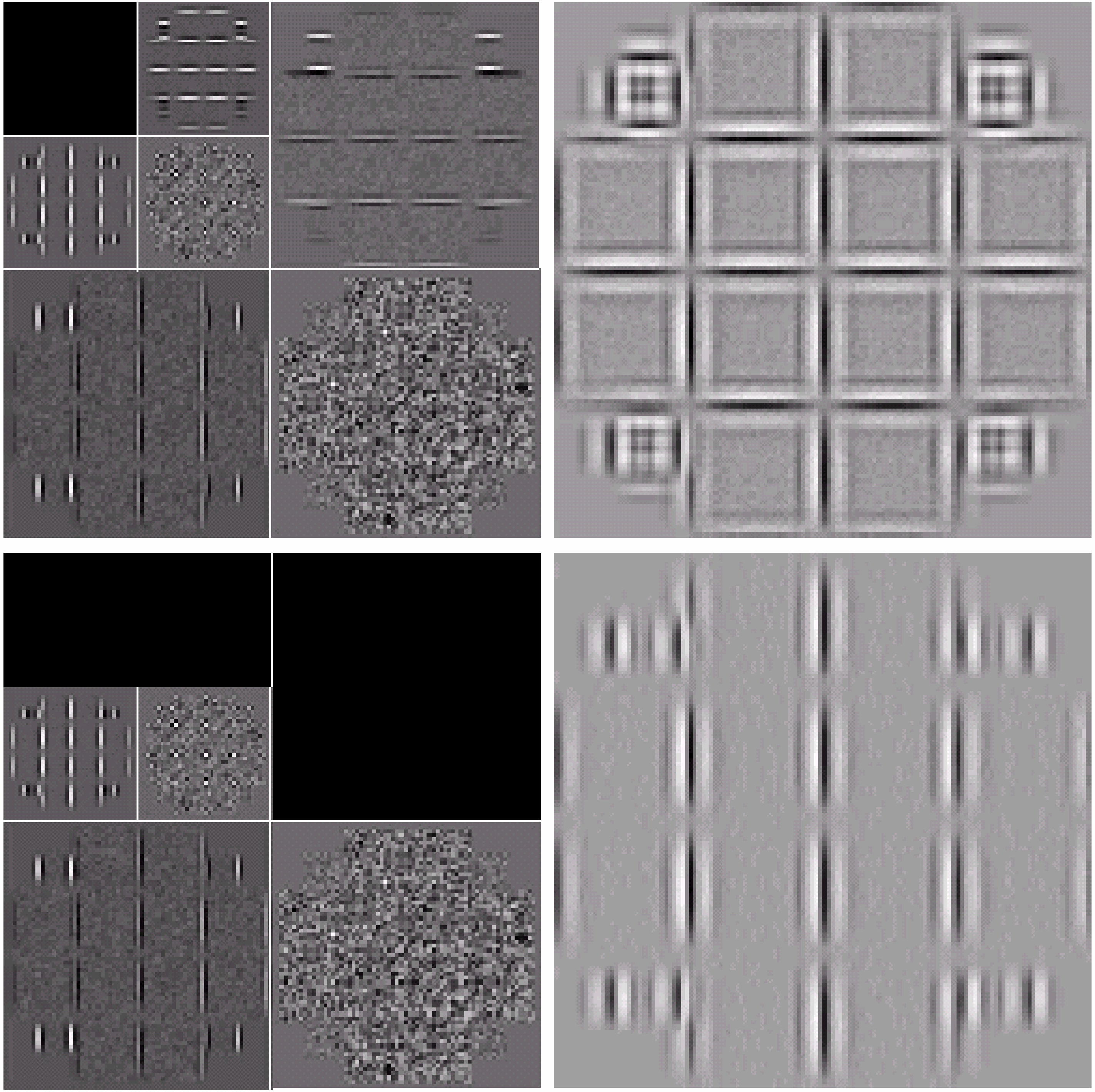
FIGURE 7.24 Fourth-order symlets: (a)–(b) decomposition filters; (c)–(d) reconstruction filters; (e) the one-dimensional wavelet; (f) the one-dimensional scaling function; and (g) one of three two-dimensional wavelets, $\psi^H(x, y)$.

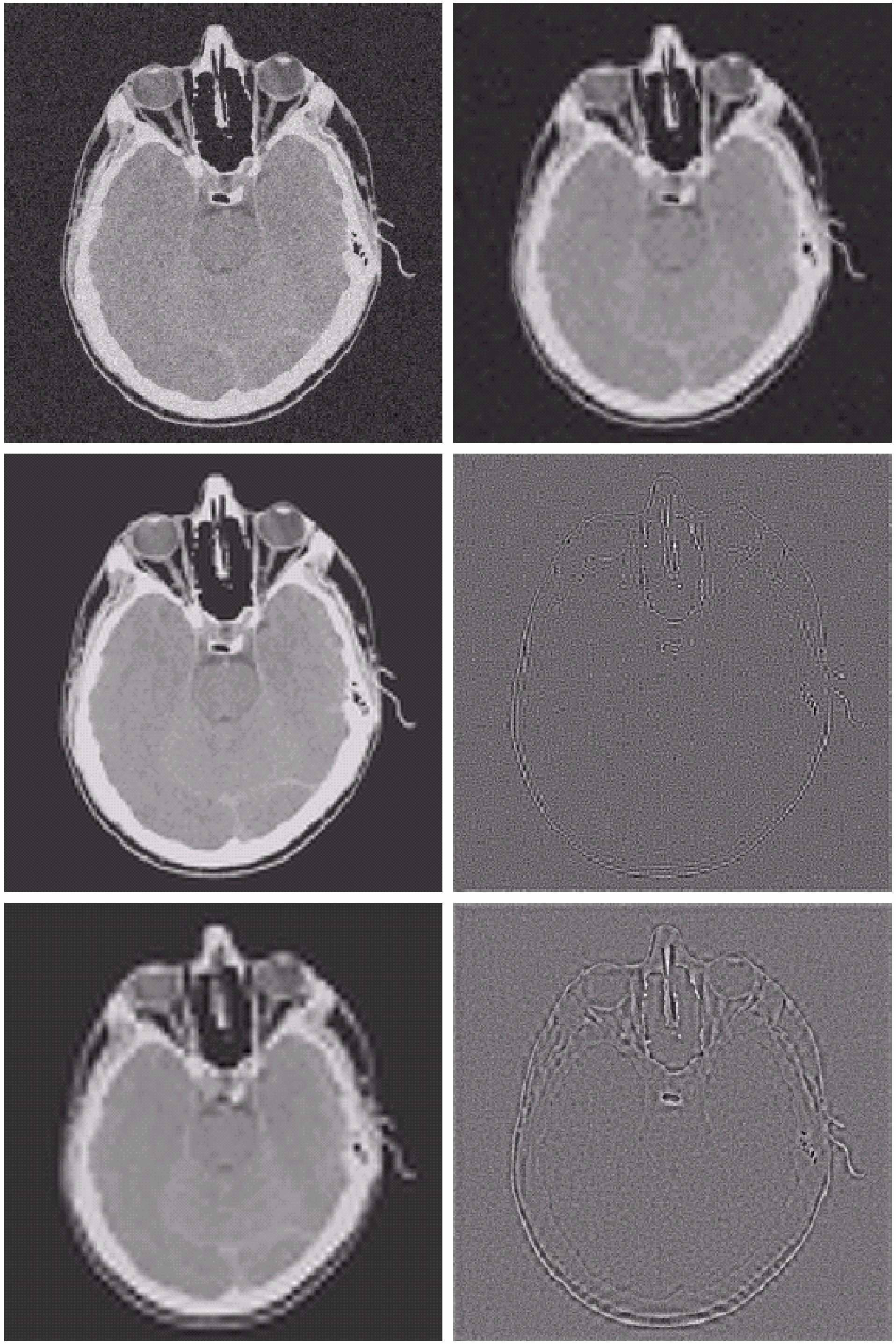
Fig. 7.24 (Con't)



a b
c d

FIGURE 7.25 Modifying a DWT for edge detection: (a) and (c) two-scale decompositions with selected coefficients deleted; (b) and (d) the corresponding reconstructions.





a b
c d
e f

FIGURE 7.26 Modifying a DWT for noise removal: (a) a noisy MRI of a human head; (b), (c) and (e) various reconstructions after thresholding the detail coefficients; (d) and (f) the information removed during the reconstruction of (c) and (e). (Original image courtesy Vanderbuilt University Medical Center.)