Dynamic programming in mathematical optimization

In terms of mathematical optimization, dynamic programming usually refers to a simplification of a decision by breaking it down into a sequence of decision steps over time. This is done by defining a sequence of **value functions** V_1, V_2, ... V_n, with an argument y representing the **state** of the system at times *i* from 1 to *n*. The definition of $V_n(y)$ is the value obtained in state *y* at the last time *n*. The values V_i at earlier times *i=n-1,n-2,...,2,1*can be found by working backwards, using a recursive relationship called the Bellman equation. For i=2,...n, V_{i-1} at any state y is calculated from $V_i$by maximizing a simple function (usually the sum) of the gain from decision *i-1* and the function V_i at the new state of the system if this decision is made. Since V_i has already been calculated, for the needed states, the above operation yields V_{i-1} for all the needed states. Finally, V_1 at the initial state of the system is the value of the optimal solution. The optimal values of the decision variables can be recovered, one by one, by tracking back the calculations already performed.

Optimal consumption and saving

A mathematical optimization problem that is often used in teaching dynamic programming to economists (because it can be solved by hand concerns a consumer who lives over the periods $t = 0,1,2,...,T$ and must decide how much to consume and how much to save in each period.

Let c_t be consumption in period *t*, and assume consumption yields utility $u(c_t) = \ln(c_t)$ as long as the consumer lives. Assume the consumer is impatient, so that he discounts future utility by a factor b each period, where $0 < b < 1$. Let k_t be capital in period *t*. Assume initial capital is a given amount $k_0 > 0$, and suppose that this period's capital and consumption determine next period's capital as $k_{t+1} = Ak_t^a - c_t$, where A is a positive constant and $0 < a < 1$. Assume capital cannot be negative. Then the consumer's decision problem can be written as follows:

$$\max \sum_{t=0}^{T} b^t \ln(c_t) \text{ subject to } k_{t+1} = Ak_t^a - c_t \geq 0 \text{ for all } t = 0,1,2,...,T$$

Written this way, the problem looks complicated, because it involves solving for all the choice variables $c_0,c_1,c_2,...,c_T$ and $k_1,k_2,k_3,...,k_{T+1}$ simultaneously. (Note that k_0 is not a choice variable—the consumer's initial capital is taken as given.)

The dynamic programming approach to solving this problem involves breaking it apart into a sequence of smaller decisions. To do so, we define a sequence of *value functions* $V_t(k)$, for$t = 0,1,2,...,T,T+1$ which represent the value of having any amount of capital k at each time t. Note that $V_{T+1}(k) = 0$, that is, there is (by assumption) no utility from having capital after death.

The value of any quantity of capital at any previous time can be calculated by backward induction using the Bellman equation. In this problem, for each $t = 0, 1, 2, ...,$ T, the Bellman equation is

$$V_t(k_t) = \max\left(\ln(c_t) + bV_{t+1}(k_{t+1})\right) \text{ subject to } k_{t+1} = Ak_t^a - c_t \geq 0$$

This problem is much simpler than the one we wrote down before, because it involves only two decision variables, c_t and k_{t+1}. Intuitively, instead of choosing his whole lifetime plan at birth, the consumer can take things one step at a time. At time t, his current capital k_t is given, and he only needs to choose current consumption c_t and saving k_{t+1}.

To actually solve this problem, we work backwards. For simplicity, the current level of capital is denoted as k. $V_{T+1}(k)$ is already known, so using the Bellman equation once we can calculate $V_T(k)$, and so on until we get to $V_0(k)$, which is the *value* of the initial decision problem for the whole lifetime. In other words, once we know $V_{T-j+1}(k)$, we can calculate $V_{T-j}(k)$, which is the maximum of $\ln(c_{T-j}) + bV_{T-j+1}(Ak^a - c_{T-j})$, where c_{T-j} is the variable and $Ak^a - c_{T-j} \geq 0$. It can be shown that the value function at time $t = T - j$ is

$$V_{T-j}(k) = a\sum_{i=0}^{j} a^i b^i \ln k + v_{T-j}$$

where each v_{T-j} is a constant, and the optimal amount to consume at time $t = T - j$ is

$$c_{T-j}(k) = \frac{1}{\sum_{i=0}^{j} a^i b^i} Ak^a$$

which can be simplified to

$$c_T(k) = Ak^a \text{, and } c_{T-1}(k) = \frac{1}{1+ab}Ak^a,$$

$$\text{and } c_{T-2}(k) = \frac{1}{1+ab+a^2b^2}Ak^a \text{, etcetera.}$$

We see that it is optimal to consume a larger fraction of current wealth as one gets older, finally consuming all current wealth in period T, the last period of life.